Random Self-reducibility of Ideal-SVP via Arakelov Random Walks

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Séminaire de théorie des nombres de Toulouse

https://eprint.iacr.org/2020/297.pdf

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Examples:

- Factoring
- Discrete logarithm

Foundation of public key cryptography

Cryptographic primitives (public key)

public key encryption | signature | homomorphic encryption | …

error correcting codes | lattices | isogenies

factoring | discrete logarithm | …

Supposedly intractable algorithmic problems

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Lattices

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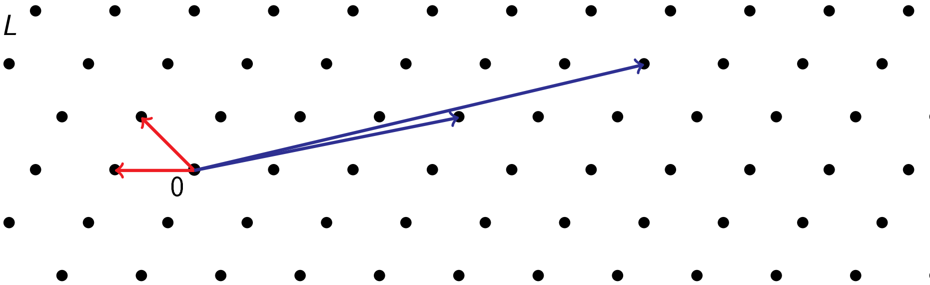
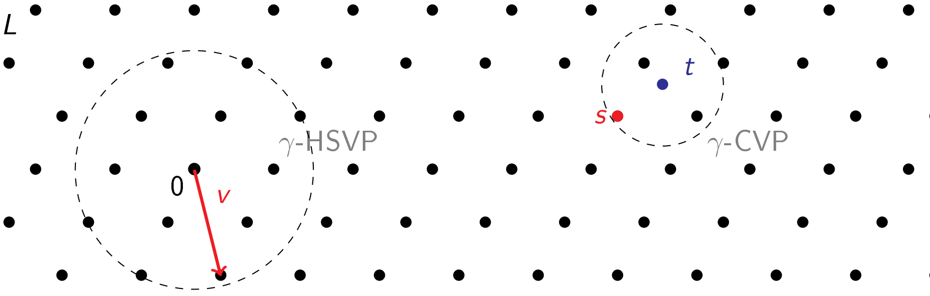


- $L = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
- $B \in \mathrm{GL}_n(\mathbb{R})$ is a basis
- n is the dimension of L
- $|\det(B)| =: \mathrm{Vol}(L)$ is the volume of L (does not depend on the basis B)
  - in this talk $\mathrm{Vol}(L) = 1$ always

Algorithmic problems



γ-HSVP

(Hermite Shortest Vector Problem)

Find $v \in L$ such that $\|v\|_2 \leq \gamma$

γ-CVP

(Closest Vector Problem)

Given $t \in \mathbb{R}^n$, find $s \in L$ such that $\|t - s\|_2 \leq \gamma$

(**input:** a basis of L)

Hardness of HSVP and CVP

γ-HSVP and γ-CVP are hard to solve

- if the input is a bad basis of L
- if $\gamma = \mathrm{poly}(n)$
- in the worst case
  - we don't have a polynomial time algorithm that works for all lattices

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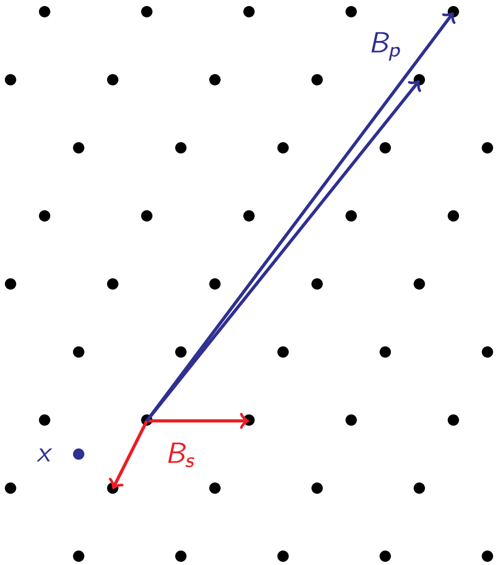
Remark: if we have a good basis of L, then they become easy

Public key encryption from lattices



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$\mathrm{sk} = B_s$

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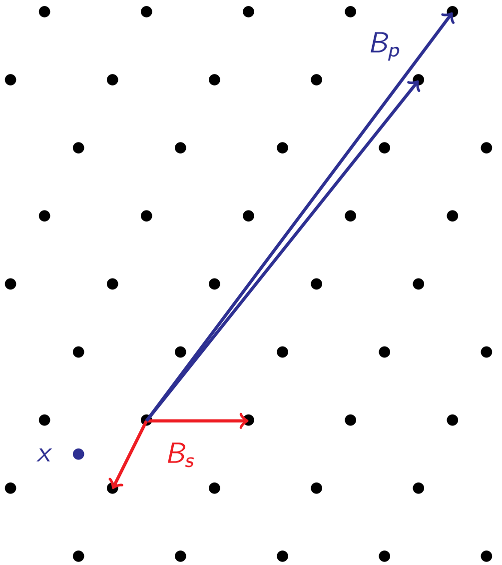

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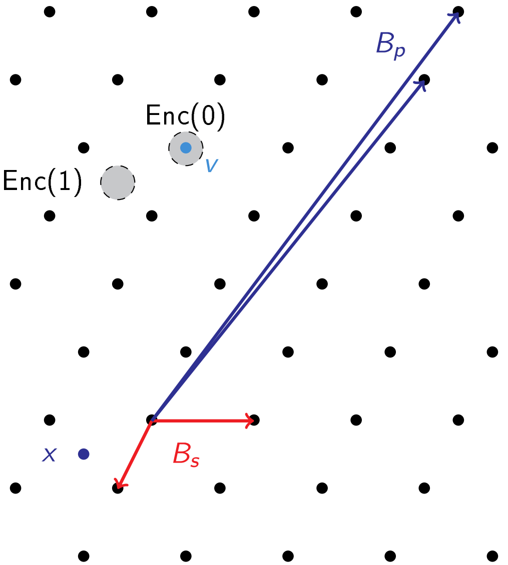

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Encryption(m, pk):

- sample random $v \in L$
- sample small $e \in \mathbb{R}^n$
- return $c = v + e + m \cdot x$

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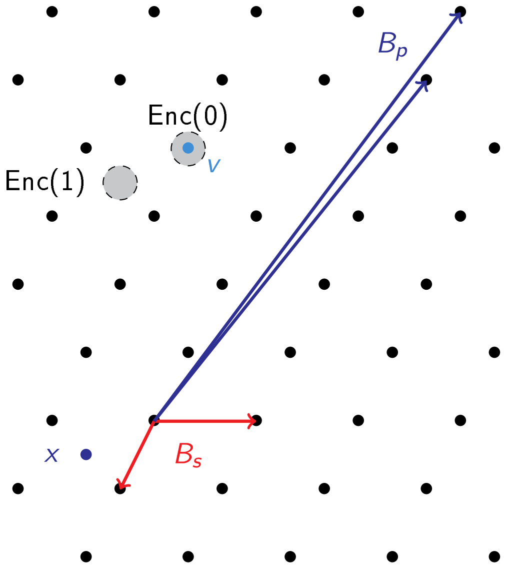

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Decryption(c, sk):

- find $w \in L$ closest to c
- if c is very close to w, return $m = 0$
- otherwise return $m = 1$

Summary (so far)

- we need hard algorithmic problems for cryptography
- γ-HSVP is such a hard problem

From now on, we focus on γ-HSVP

γ-HSVP: given a bad basis of a lattice L (with $\mathrm{vol}(L) = 1$), find $v \in L$ such that $\|v\|_2 \leq \gamma$

Ideal lattices

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Schemes using lattices are usually not efficient
(storage: n^2, matrix-vector mult: n^2)
$\Rightarrow$ improve efficiency using ideal lattices

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$$M_a = \begin{pmatrix} a_1 & -a_n & \cdots & -a_2 \\ a_2 & a_1 & \cdots & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \dots & a_1 \end{pmatrix}$$

basis of a special case of
ideal lattice

Some definitions

Notation

$K = \mathbb{Q}[X]/(X^n+1)$, with $n = 2^k$ (or any number field)

$O_K = \mathbb{Z}[X]/(X^n+1)$

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- integral ideal: $\mathfrak{a} \subseteq O_K$
- oriented replete ideal: $I := \alpha \cdot \mathfrak{a} \subset K_{\mathbb{R}}$, with $\alpha \in K_{\mathbb{R}}$ and $\mathfrak{a} \subseteq O_K$
  (e.g., $I = \sqrt{2} \cdot \langle 3 \rangle = \{\sqrt{2} \cdot 3 \cdot x \mid x \in \mathbb{Z}\} \subset \mathbb{R}$)

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From now on:

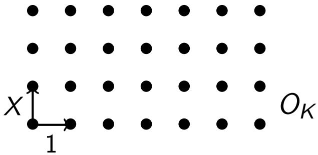
- ideal := oriented replete ideal
- I is an ideal
- $\mathcal{N}(I) = 1$

Why is I a lattice?

O_K is a lattice

$$\begin{aligned} \sigma : O_K = \mathbb{Z}[X]/(X^n+1) &\to \mathbb{C}^n \\ r(X) &\mapsto (r(\alpha_1), r(\alpha_2), \dots, r(\alpha_n)), \end{aligned}$$

where $\alpha_1, \dots, \alpha_n$ are the roots of $X^n + 1$ in $\mathbb{C}$



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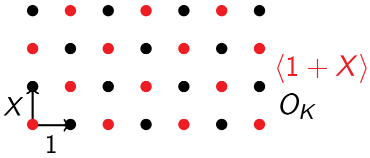
$$\begin{cases} \sigma(I) \subseteq \sigma(O_K) \simeq \mathbb{Z}^n \\ \text{stable by '+' and '−'} \end{cases} \Rightarrow \text{ideal lattice}$$



γ-ideal-HSVP

γ-ideal-HSVP = γ-HSVP restricted to ideal lattices

γ-ideal-HSVP: given a basis of an ideal lattice $\sigma(I)$ (with $\mathcal{N}(I) = 1$) , find $x \in I$ such that $\|\sigma(x)\|_2 \leq \gamma$.

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This is still a hard problem

- if the input basis of $\sigma(I)$ is bad
- if $\gamma = \mathrm{poly}(d)$
- in the worst case

  (no poly time algorithm that works for all ideal lattices)

Summary (so far)

- we need hard algorithmic problems for cryptography
- γ-HSVP is a hard problem
- γ-HSVP restricted to ideal lattices is still a hard problem

From now on, we focus on γ-ideal-HSVP

γ-ideal-HSVP: given a bad basis of an ideal lattice $\sigma(I)$ (with $\mathcal{N}(I) = 1$) , find $x \in I$ such that $\|\sigma(x)\|_2 \leq \gamma$.

Average-case hardness

Worst-case hardness

γ-ideal-HSVP is hard in the worst case:

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How do we generate ideals I for which γ-ideal-HSVP is hard?

(this is needed for crypto)

Our result

Theorem [BDPW20]

There is a distribution D over ideal lattices such that

solving γ-ideal-HSVP in I with non-negligible probability when $I \leftarrow D$
$\Rightarrow$ solving γ'-ideal-HSVP in all ideals I

with $\gamma' = \sqrt{d} \cdot \gamma$

γ-ideal-HSVP is hard on average.

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We can sample hard ideal lattices for crypto

(very small probability that the sampled ideal is an easy one)

Techniques of the proof

Conclusion

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[Gen10] Gentry. Toward basing fully homomorphic encryption on worst-case hardness. Crypto

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- ideal-HSVP is believed to be hard in the worst case
- we show that if ideal-HSVP is hard in the worst-case, then it is also hard on average.
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- a worst-case to average-case reduction was already proven in [Gen10]
  - requires a quantum computer
  - worse loss $\gamma \to \gamma'$
  - different distribution D and different proof

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Thank you

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