On the hardness of the NTRU problem

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Séminaire de cryptographie de Rennes

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NTRU

Algorithmic problem based on lattices

Definition (informal)

An NTRU instance is

$$h = f \cdot g^{-1} \bmod q,$$

where $f, g \in \mathbb{Z}$ and $|f|, |g| \ll \sqrt{q}$.

Decision-NTRU: Distinguish $h = f \cdot g^{-1} \bmod q$ from h uniform

Search-NTRU: Recover (f, g) from h.

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- post-quantum
- efficient
- used in Falcon and NTRU / NTRUPrime (NIST finalists)
- old (for lattice-based crypto)

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RLWE

Another lattice-based algorithmic problem

Definition (informal)

A RLWE instance is

$$(a_i, b_i = a_i \cdot s + e_i \bmod q)_{1 \le i \le m},$$

with a uniform in $\mathbb{Z}/(q\mathbb{Z})$ and $s, e \in \mathbb{Z}$ such that $|s|, |e| \ll \sqrt{q}$.

Decision-RLWE: Distinguish $b_i = a_i \cdot s + e_i$ from b_i uniform (a_i public)

Search-RLWE: Recover s from $(a_i, b_i)_i$.

[SSTX09] Stehlé, Steinfeld, Tanaka, and Xagawa. Efficient public key encryption based on ideal lattices. Asiacrypt.

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NTRU vs RLWE

- both are efficient
- both are versatile (but RLWE a bit more)
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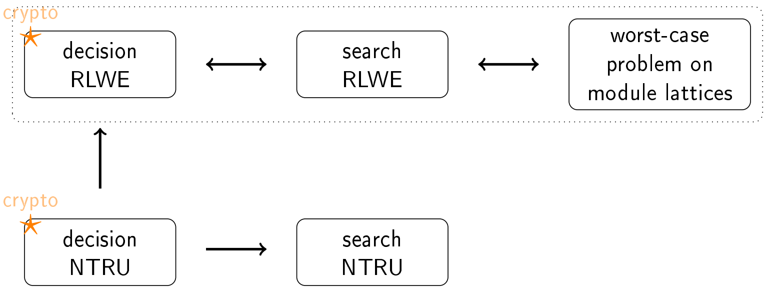
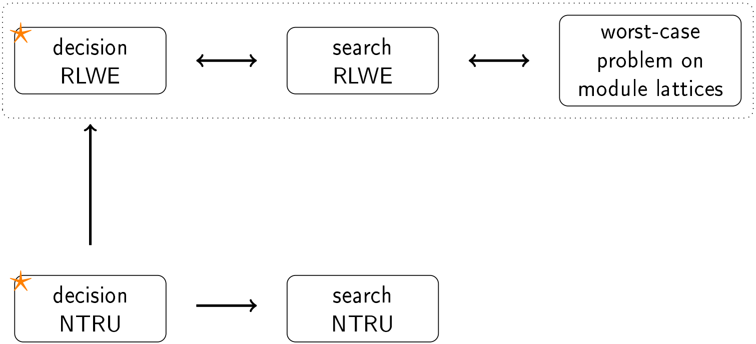
- both are efficient
- both are versatile (but RLWE a bit more)
- NTRU is older
- RLWE has better security guarantees



Our result



decision RLWE
search RLWE
worst-case problem on module lattices
decision NTRU
search NTRU

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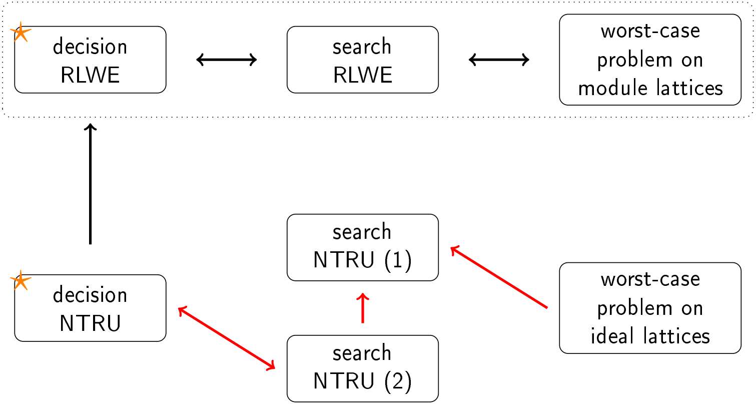
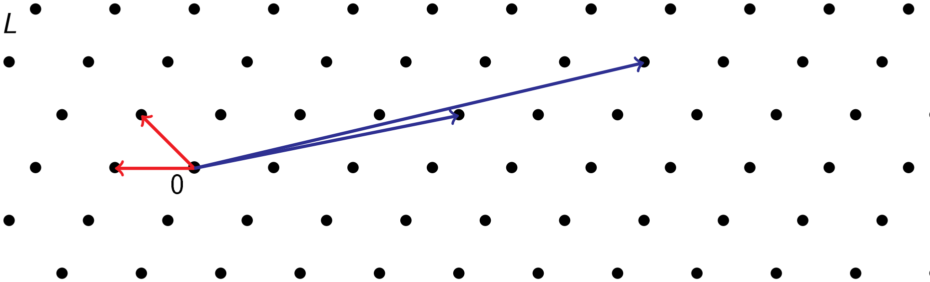

decision RLWE
search RLWE
worst-case problem on module lattices
search NTRU (1)
decision NTRU
search NTRU (2)
worst-case problem on ideal lattices

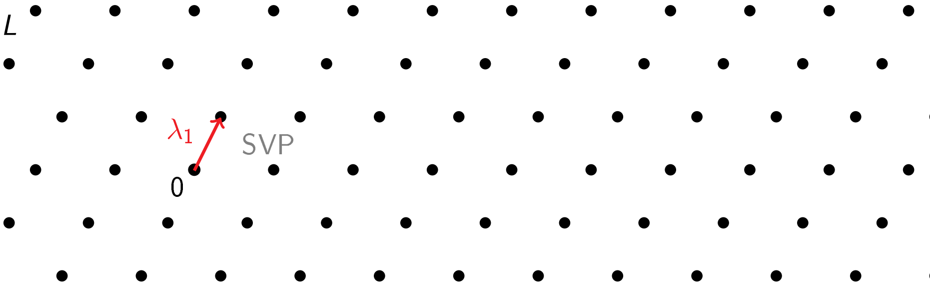
Lattices and ideals

Lattices



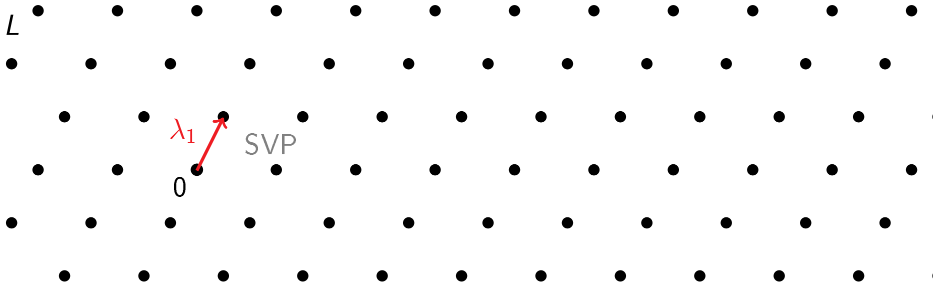
- $L = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
- $B \in \mathrm{GL}_n(\mathbb{R})$ is a basis
- n is the dimension of L

Shortest vector problem



SVP : Shortest Vector Problem

Shortest vector problem

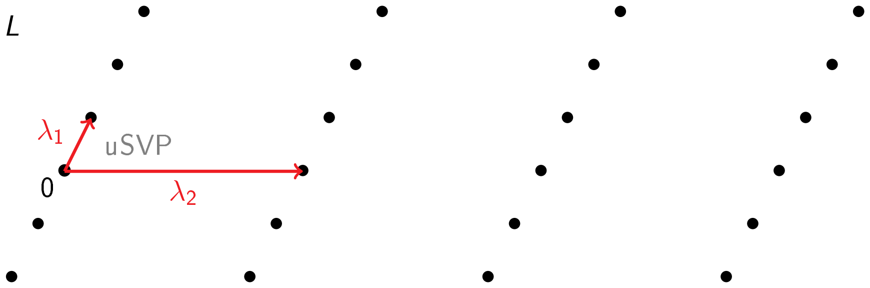


SVP : Shortest Vector Problem

Supposedly hard to solve when n is large

- even with a quantum computer
- even for some structured lattices (e.g., ideal lattices)

Unique shortest vector problem



uSVP : unique Shortest Vector Problem

uSVP = SVP restricted to lattices with $\lambda_1 \ll \lambda_2$

Ideal lattices

- $R = \mathbb{Z}[X]/(X^n + 1)$ with $n = 2^k$ (or $R = \mathbb{Z}$)
- $K = \mathbb{Q}[X]/(X^n + 1)$ (or $K = \mathbb{Q}$)

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$$\sigma : \quad K = \mathbb{Q}[X]/(X^n+1) \to \mathbb{Q}^n$$

$$r = \sum_{i=0}^{n-1} r_i X^i \mapsto (r_0, \cdots, r_{n-1})$$

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$\sigma(\langle 2 \rangle) \subset \mathbb{Q}$

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Ideal lattice: $\sigma(\langle z \rangle) \subset \mathbb{Q}^n$ is a lattice

ideal-SVP: Given $\langle z \rangle$, find $rz \in \langle z \rangle$ such that $\|\sigma(rz)\|$ is small

The different NTRU problems

NTRU instances

$R_q := R/(qR)$

NTRU instance

A (γ, q)-NTRU instance is $h \in R_q$ s.t.

- $h = f/g \bmod q$ (or $gh = f \bmod q$)
- $\|f\|, \|g\| \leq \frac{\sqrt{q}}{\gamma}$ (if $y = \sum_{i=0}^{n-1} y_i X^i \in R$, then $\|y\| = \sqrt{\sum_i y_i^2}$)

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The pair (f, g) is a trapdoor for h.

Claim: if (f, g) and (f', g') are two trapdoors for the same h,

$$\frac{f'}{g'} = \frac{f}{g} =: h_K \in K$$ (division performed in K)

Decisional NTRU problem

dNTRU

The (γ, q)-decisional NTRU problem ((γ, q)-dNTRU) asks, given $h \in R_q$, to decide whether

- $h \leftarrow \mathcal{D}$ where $\mathcal{D}$ is a distribution over (γ, q)-NTRU instances
- $h \leftarrow \mathcal{U}(R_q)$

Search NTRU problems

$\mathrm{NTRU}_{\mathrm{vec}}$

The (γ, q)-search NTRU vector problem ((γ, q)-$\mathrm{NTRU}_{\mathrm{vec}}$) asks, given a (γ, q)-NTRU instance h, to recover $(f, g) \in R^2$ s.t.

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(Recall $h_K = f/g \in K$ for any trapdoor (f, g))
$\Leftrightarrow$ recover $(\alpha f, \alpha g)$ for any $\alpha \in K$

(The two problems exist in worst-case and average-case variants)

NTRU is a (module) lattice problem

NTRU lattice: For $h \in R$, define Λ_h the (module) lattice with basis

$$B_h = \begin{pmatrix} 1 & 0 \\ h & q \end{pmatrix} \quad \text{(in columns)}$$

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$\mathrm{NTRU}_{\mathrm{vec}}$: recover $(f, g) \leftrightarrow$ find a shortest vector in Λ_h

$\mathrm{NTRU}_{\mathrm{mod}}$: recover $\alpha \cdot (f, g) \leftrightarrow$ find the direction where Λ_h is dense

What we know about NTRU

Previous works

Reductions:

[SS11, WW18] If $f, g \leftarrow D_{R,\sigma}$ with $\sigma \geq \mathrm{poly}(n) \cdot \sqrt{q}$
then $f/g \approx \mathcal{U}(R_q)$ (cyclotomic fields)

- dNTRU is provably hard when $\gamma \leq \frac{1}{\mathrm{poly}(n)}$

[SS11] Stehlé and Steinfeld. Making NTRU as secure as worst-case problems over ideal lattices. Eurocrypt.

[WW18] Wang and Wang. Provably secure NTRUEncrypt over any cyclotomic field. SAC.

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[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

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Attacks: (polynomial time)

[LLL82] dNTRU, $\mathrm{NTRU}_{\mathrm{mod}}$ broken if $\gamma \geq 2^n$

[LLL82] Lenstra, Lenstra, Lovász. Factoring polynomials with rational coefficients. Mathematische Annalen.

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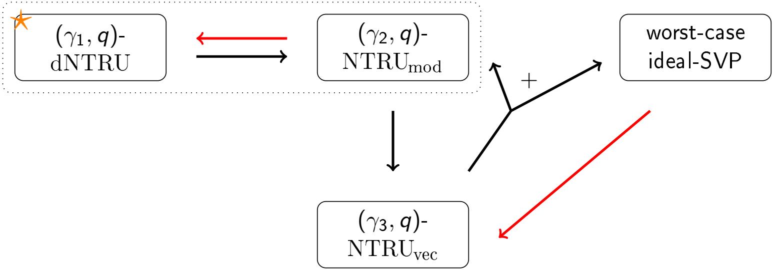
[ABD16, CJL16] [KF17] dNTRU, $\mathrm{NTRU}_{\mathrm{mod}}$ broken if $(\log q)^2 \geq n \cdot \log \frac{\sqrt{q}}{\gamma}$
(e.g., $q \approx 2^{\sqrt{n}}$ and $\gamma = \sqrt{q}/\mathrm{poly}(n)$)

[ABD16] Albrecht, Bai, and Ducas. A subfield lattice attack on overstretched NTRU assumptions. Crypto.

[CJL16] Cheon, Jeong, and Lee. An algorithm for NTRU problems. LMS J Comput Math.

[KF17] Kirchner and Fouque. Revisiting lattice attacks on overstretched NTRU parameters. Eurocrypt

Our results (with more details)



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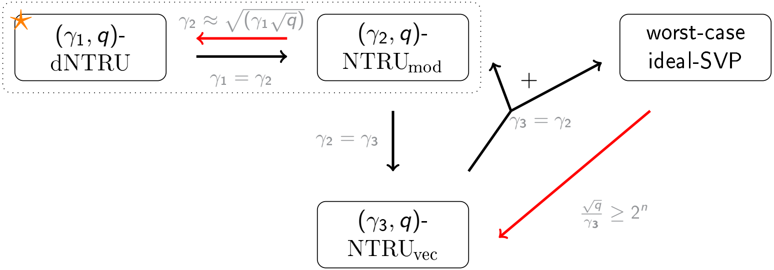

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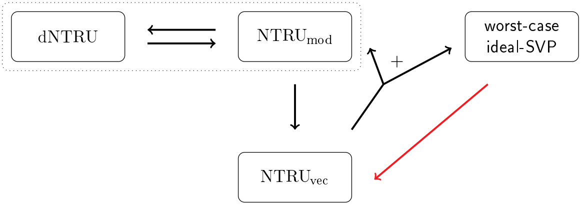
Techniques


dNTRU
$\mathrm{NTRU}_{\mathrm{mod}}$
worst-case
ideal-SVP
+
$\mathrm{NTRU}_{\mathrm{vec}}$

From ideal-SVP to $\mathrm{NTRU}_{\mathrm{vec}}$

Objective: Transform an ideal I into an NTRU instance h

- $I = \langle z \rangle = \{z \cdot r \mid r \in R\}$
- g short vector of I

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\Leftrightarrow\ & g \cdot \frac{q}{z} = qr \\
\Leftrightarrow\ & g \cdot h = f \bmod q
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⚠ Not an NTRU instance ($h \in K$ is not in R_q)

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This is an NTRU instance

From ideal-SVP to $\mathrm{NTRU}_{\mathrm{vec}}$ (2)

Summing up: If $I = \langle z \rangle = \{z \cdot r \mid r \in R\}$ and z known

- can construct an NTRU instance h from I
  - any short $g \in I$ provides a trapdoor (f, g) for h

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What we need to conclude the reduction:

- any trapdoor (f', g') for h is such that $g' \in I$
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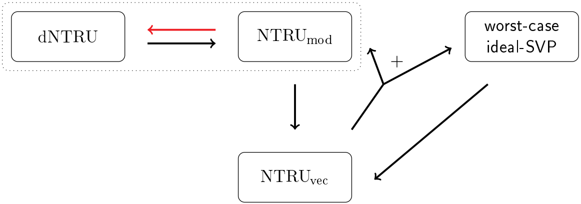
And for non principal ideals?

- $I = R \cap \langle z \rangle$ and z easily computed
  - everything still works with this z

Techniques



From $\mathrm{NTRU}_{\mathrm{mod}}$ to dNTRU

Objective: given $h = f/g \bmod q$, recover $h_K = f/g \in K$ (division in K)

Can use an oracle: given $h \in R_q$, outputs

- YES if $h = f/g \bmod q$, with f, g small $(\leq B)$
- NO otherwise

From $\mathrm{NTRU}_{\mathrm{mod}}$ to dNTRU

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Can use an oracle: given $h \in R_q$, outputs

- YES if $h = f/g \bmod q$, with f, g small ($\leq B$)
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Idea:

- take $x, y \in R$
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$\Rightarrow$ we can choose x and y

$\Rightarrow$ we can modify the coordinates one by one

From $\mathrm{NTRU}_{\mathrm{mod}}$ to dNTRU (2)

Simplified problem

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- Solve for f/g

More technical details

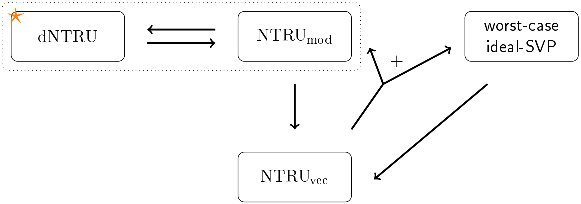
We do not have a perfect oracle

- need to handle distributions
- use the "oracle hidden center" framework [PRS17]

[PRS17] Peikert, Regev, and Stephens-Davidowitz. Pseudorandomness of ring-LWE for any ring and modulus. STOC.

Conclusion

Conclusion and open problems



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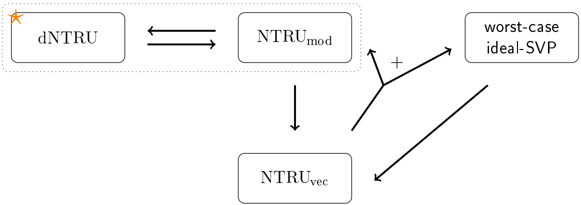

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Thank you