# Approx-SVP in Ideal lattices with Pre-Processing

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LIP, ENS de Lyon

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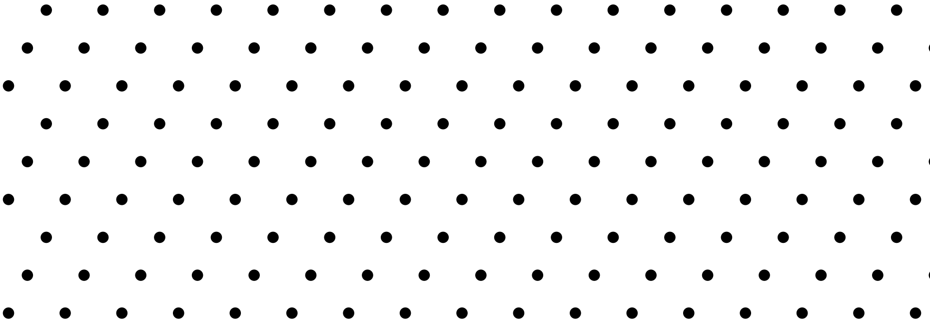
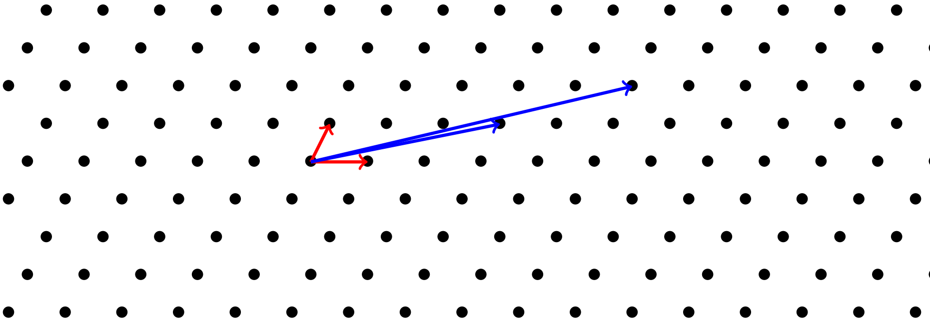
# Lattices



## Lattice

A lattice $L$ is a 'vector space' over $\mathbb{Z}$.

# Lattices

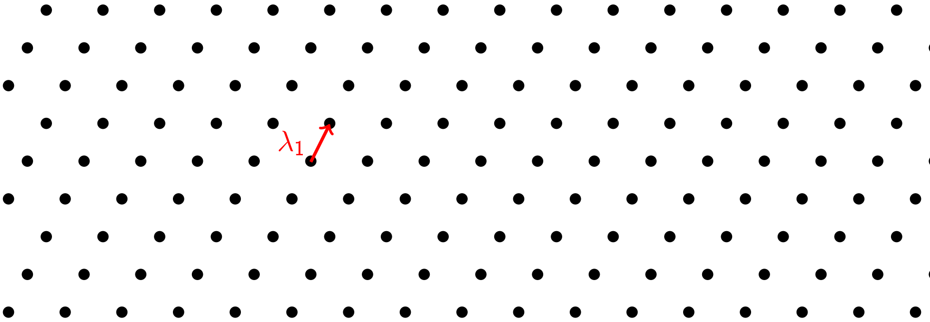


## Lattice

A lattice $L$ is a 'vector space' over $\mathbb{Z}$.
A basis of $L$ is an invertible matrix $B$ such that $L = \{Bx \mid x \in \mathbb{Z}^n\}$.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 10 \\ 4 & 2 \end{pmatrix}$ are two basis of the above lattice.

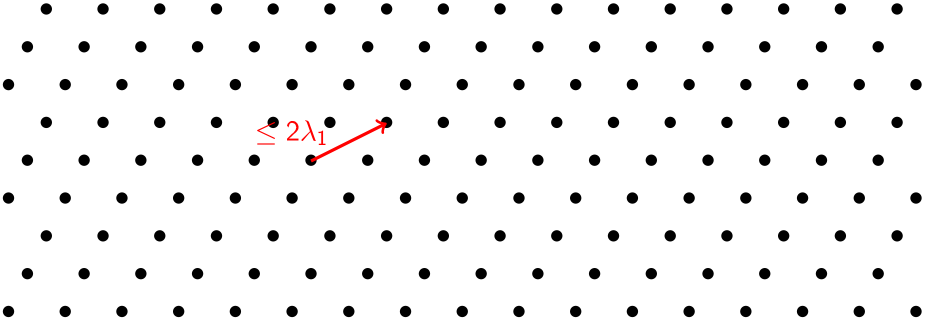
# Lattices




**Shortest Vector Problem (SVP)**

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted $\lambda_1$.

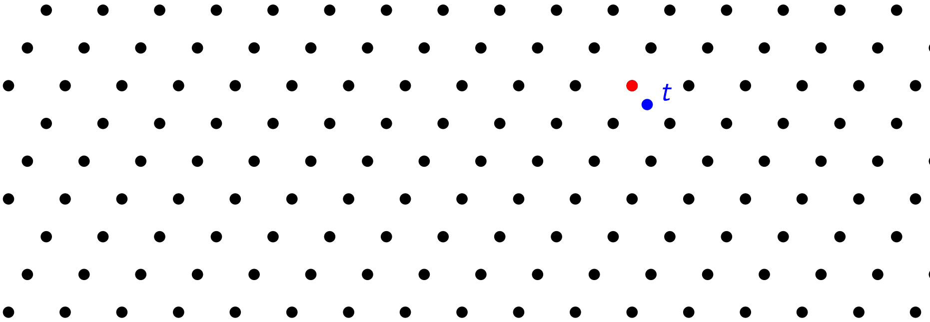
# Lattices




**Approximate Shortest Vector Problem (approx-SVP)**

Find a short (in Euclidean norm) non-zero vector.
(of norm $\leq 2\lambda_1$ for instance).

# Lattices




**Closest Vector Problem (CVP)**

Given a target point $t$, find a point of the lattice closest to $t$.

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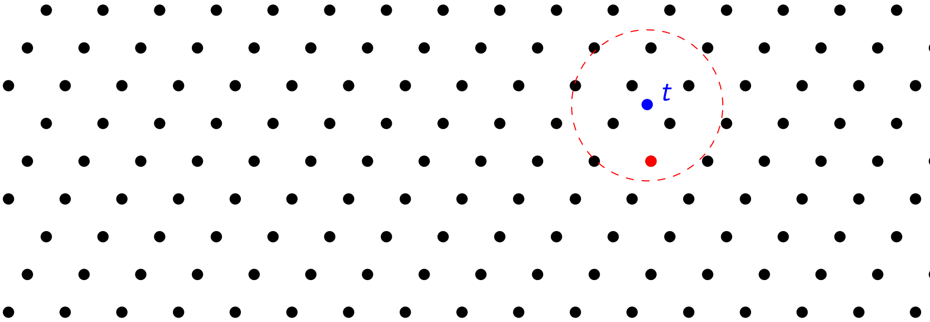


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# Complexity of SVP/CVP

## Applications

SVP and CVP in general lattices are conjectured to be hard to solve both quantumly and classically $\Rightarrow$ used in cryptography

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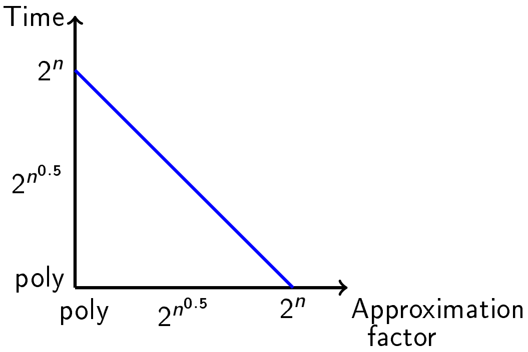
Best Time/Approximation trade-off for general lattices: BKZ algorithm



# Structured lattices

Improve efficiency of lattice-based crypto using structured lattices.

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The Ring Learning with Error (RLWE) problem is at least as hard as approx-SVP in ideal lattices.

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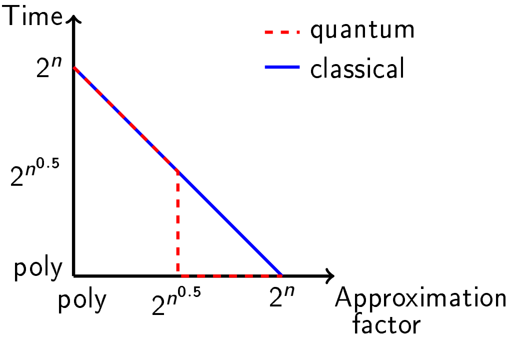
The Ring Learning with Error (RLWE) problem is at least as hard as approx-SVP in ideal lattices.

Many cryptographic constructions based on RLWE.

*Is approx-SVP still hard when restricted to ideal lattices?*

# SVP in ideal lattices

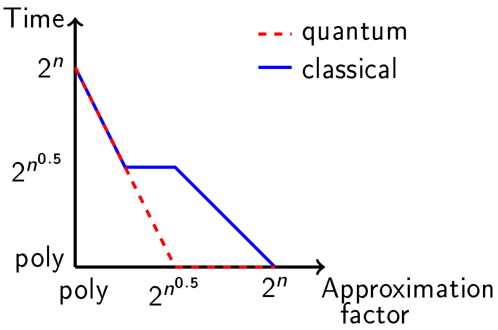
[CDPR16,CDW17]: Better than BKZ in the quantum setting




[CDPR16] R. Cramer, L. Ducas, C. Peikert and O. Regev. Recovering Short Generators of Principal Ideals in Cyclotomic Rings. Eurocrypt 2016.

[CDW17] R. Cramer, L. Ducas, B. Wesolowski. Short Stickelberger Class Relations and Application to Ideal-SVP. Eurocrypt 2017.

# This work




- Heuristic
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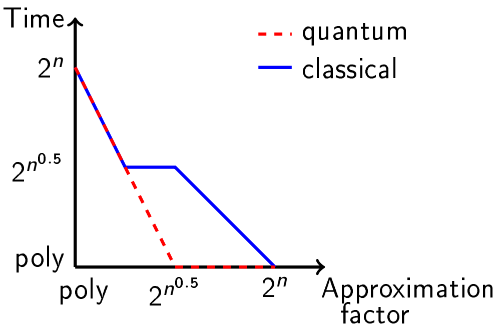


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**Disclaimer:** In this talk, only *principal* ideal lattices

# Outline of the talk

1. Definitions and objective
2. The CDPR algorithm
3. This work

# First definitions

## Notation

$R = \mathbb{Z}[X]/(X^n + 1)$ for $n = 2^k$

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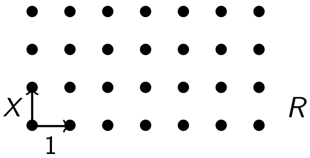
- Units: $R^\times = \{a \in R \mid \exists b \in R, ab = 1\}$
  - E.g. $\mathbb{Z}^\times = \{1, -1\}$.
- Principal ideals: $\langle g \rangle = \{gr \mid r \in R\}$ (i.e. all multiples of $g$)
  - $g$ is called a generator of $\langle g \rangle$
  - The generators of $\langle g \rangle$ are exactly the $ug$ for $u \in R^\times$
  - E.g. in $\mathbb{Z}$: $\langle 2 \rangle = \{\text{even numbers}\} = \langle -2 \rangle$

# Geometric structure

For all $r \in R$, $r = r_0 + r_1 X + \cdots + r_{n-1} X^{n-1}$, with $r_i \in \mathbb{Z}$.

- Euclidean norm: $\|r\| = \sqrt{\sum_{i=0}^{n-1} r_i^2}$.
- $R \cong \mathbb{Z}^n$ is a lattice.



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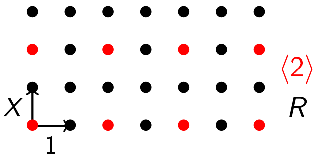
- $\langle g \rangle$ is a sub-lattice of $R$.
  - E.g. $\langle 2 \rangle \cong (2\mathbb{Z})^n$.



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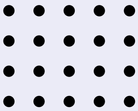
## Minkowski's embedding

- $\zeta \in \mathbb{C}$ primitive $2n$-th root of unity ($\zeta^{2n} = 1$)
- $\sigma(r) = (r(\zeta), r(\zeta^3), \cdots, r(\zeta^{n-1})) \in \mathbb{C}^{n/2} \cong \mathbb{R}^n$
- $R \mapsto \sigma(R)$ preserves the geometry (isometry + scaling)



$R$

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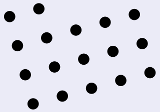

$\sigma(R)$

# Algebraic structure

## Notation

$$\sigma(r) = (\widetilde{r_1}, \cdots, \widetilde{r_{n/2}}) \in \mathbb{C}^{n/2}$$

- Algebraic norm: $\mathcal{N}(r) = \prod_{i=1}^{n/2} |\widetilde{r_i}|^2 \in \mathbb{R}$.
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- Properties:
  - $\mathcal{N}(ab) = \mathcal{N}(a) \cdot \mathcal{N}(b)$ for all $a, b \in R$,
  - $\mathcal{N}(a) \geq 1$ and $\mathcal{N}(a) \in \mathbb{Z}$ for all $a \in R \setminus \{0\}$,
  - $\mathcal{N}(u) = 1 \iff u \in R^\times$.

# Relations between algebraic/geometric structures

**Reminder:** $\sigma(r) = (\widetilde{r_1}, \cdots, \widetilde{r_{n/2}})$

- $\|r\| = \sqrt{\sum_i |\widetilde{r_i}|^2}$
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- Euclidean/algebraic norm:
  - $\|r\|$ small $\Rightarrow \mathcal{N}(r)$ relatively small.
  - $\mathcal{N}(r)$ small $\not\Rightarrow \|r\|$ relatively small (e.g. $(2^{-50}, 2^{50})$).

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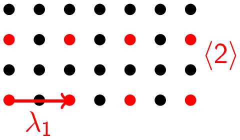
- $\lambda_1(\langle g \rangle) = \mathrm{poly}(n) \cdot \mathcal{N}(g)^{1/n}$



# Objective of this talk

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Given a basis of a principal ideal $\langle g \rangle$ and $\alpha \in (0,1]$,
Find $r \in \langle g \rangle$ such that $\|r\| \leq 2^{\widetilde{O}(n^\alpha)} \cdot \lambda_1 = 2^{\widetilde{O}(n^\alpha)} \cdot \mathcal{N}(g)^{1/n}$.

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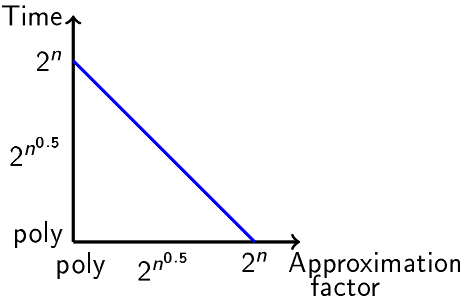
BKZ algorithm can do it in time $2^{\tilde{O}(n^{1-\alpha})}$, can we do better?



# Outline of the talk

# Overview of the CDPR algorithm (on an idea of [CGS14])

## Important points

- Large algebraic norm $\Rightarrow$ large Euclidean norm.
- In $\langle g \rangle$, the elements with the smallest algebraic norm are the generators.

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[CGS14]: P. Campbell, M. Groves, and D. Shepherd. Soliloquy: A cautionary tale.

[BS16]: J.F. Biasse, F. Song. Efficient quantum algorithms for computing class groups and solving the principal ideal problem in arbitrary degree number fields, SODA.

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**The CDPR algorithm:** find a generator with a smallest Euclidean norm

- Find a generator $g_1$ of $\langle g \rangle$
  - [BS16]: quantum time $\mathrm{poly}(n)$
  - [BEFGK17]: classical time $2^{\widetilde{O}(\sqrt{n})}$
- Find $u \in R^\times$ which minimizes $\|ug_1\|$.

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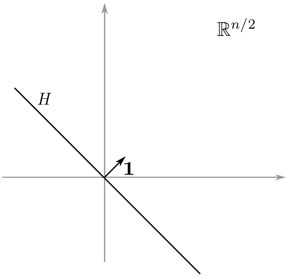
# The Log unit lattice

## Definitions

$$\begin{aligned} \mathrm{Log} : \sigma(R) &\to \mathbb{R}^{n/2} \\ (\widetilde{r_1}, \cdots, \widetilde{r_{n/2}}) &\mapsto (\log|\widetilde{r_1}|, \cdots, \log|\widetilde{r_{n/2}}|) \end{aligned}$$

Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^{\perp}$.



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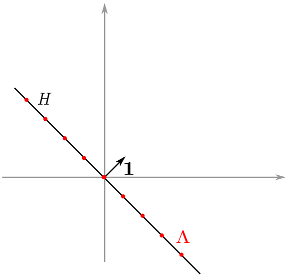
**Theorem (Dirichlet)**

$\Lambda := \mathrm{Log}(R^{\times})$ is a lattice included in $H$.



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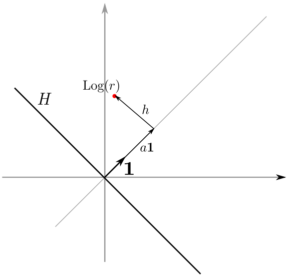


Write $\boxed{\mathrm{Log}(r) = h + a\mathbf{1}}$, with $h \in H$

- $\|r\| \le \sqrt{n} \cdot 2^{a} \cdot 2^{\|h\|}$
- $a = \frac{\log|\mathcal{N}(r)|}{n}$

# CDPR (upper bound)

## Reminder ($\mathrm{Log}(r) = h + a\mathbf{1}$)

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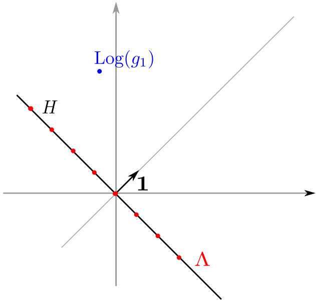


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  - quantum poly time [BS16]

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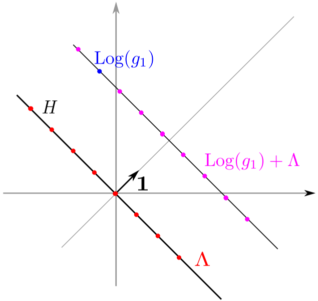


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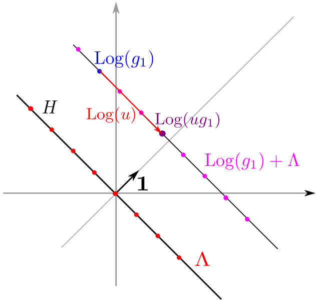


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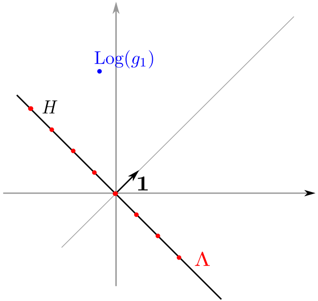


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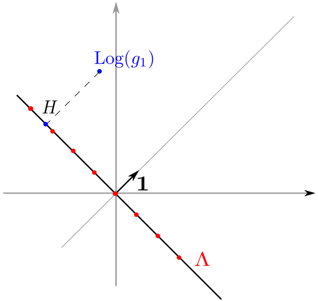


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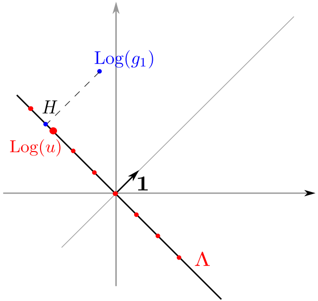


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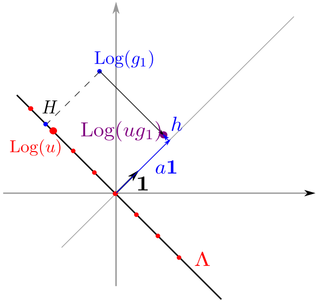


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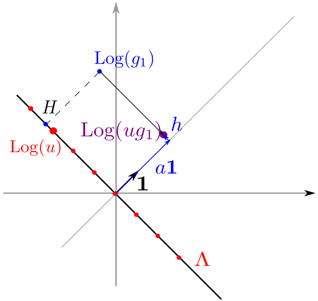


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- Solve CVP in $\Lambda$.
  - Good basis of $\Lambda$
    $\Rightarrow$ CVP in poly time
    $\Rightarrow \|h\| \leq \widetilde{O}(\sqrt{n})$

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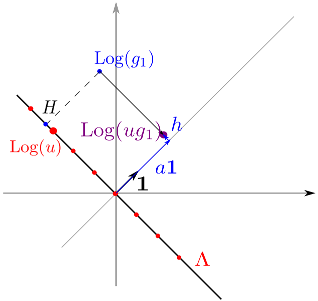


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**Reminder ($\mathrm{Log}(r) = h + a\mathbf{1}$)**

- $\|r\| \leq \sqrt{n} \cdot 2^a \cdot 2^{\|h\|}$
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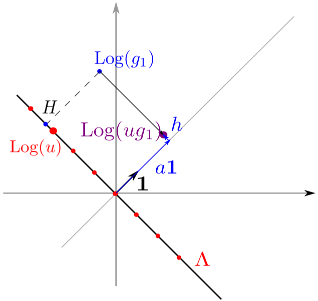


**The CDPR Algorithm:**

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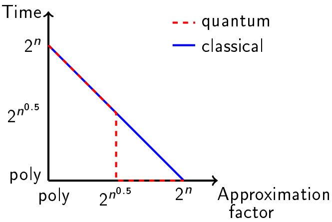


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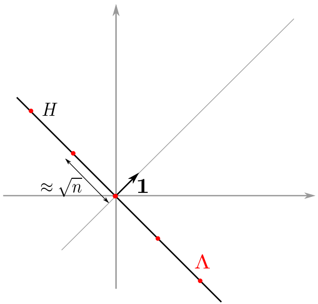


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There exists $t \in H$ such that

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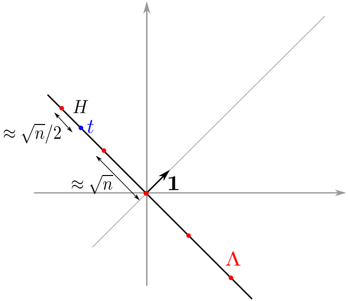


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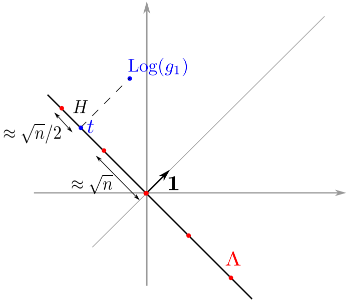


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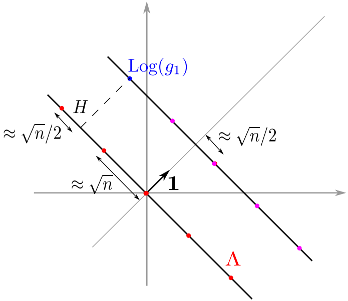


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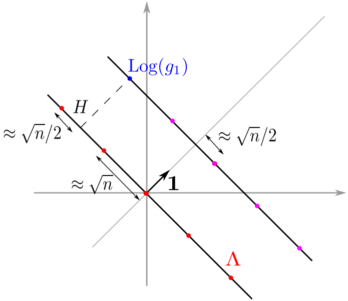


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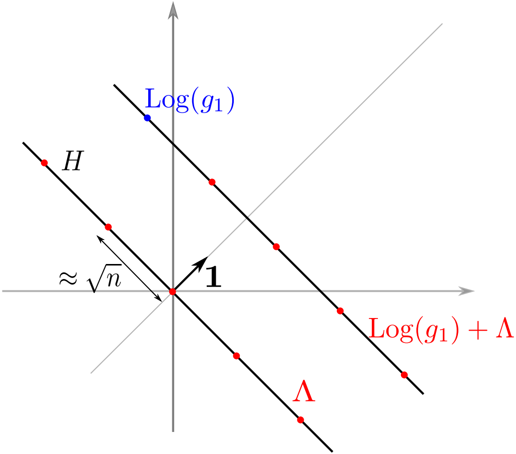
# Outline of the talk

# Idea



$\mathrm{Log}(g_1)$
$H$
$\approx \sqrt{n}$
$\mathbf{1}$
$\mathrm{Log}(g_1) + \Lambda$
$\Lambda$

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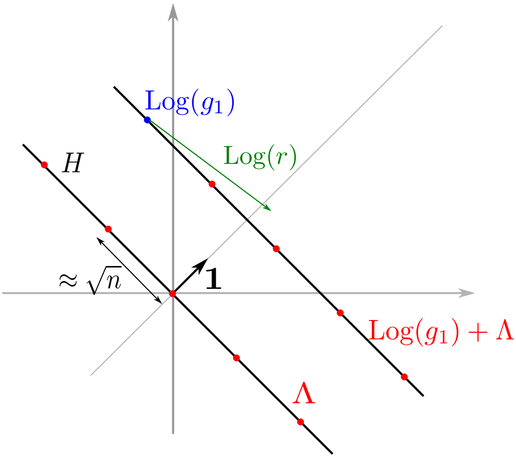

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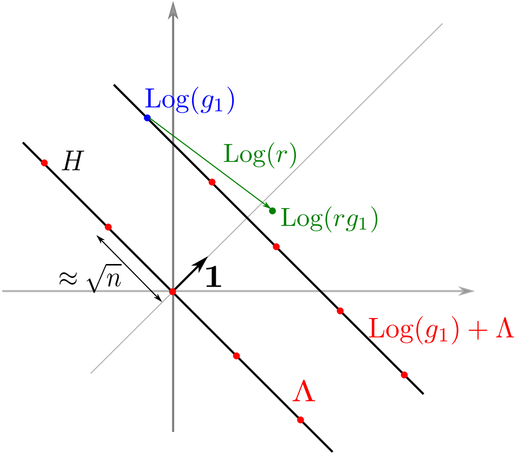

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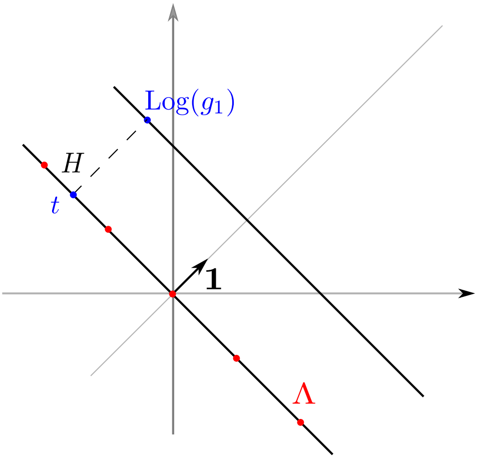

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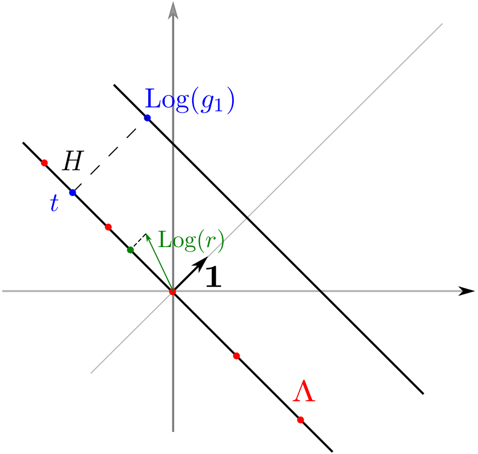

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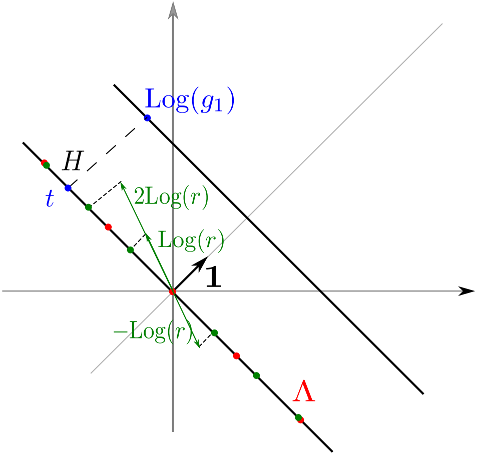

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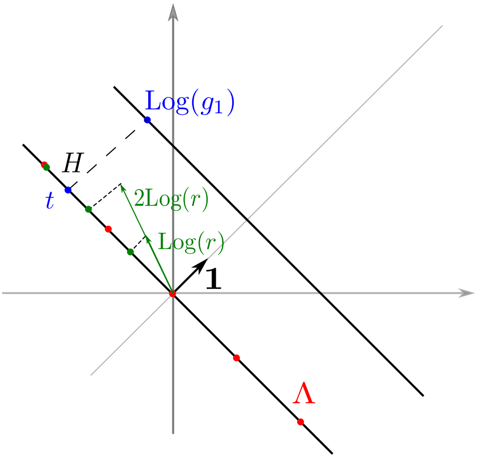

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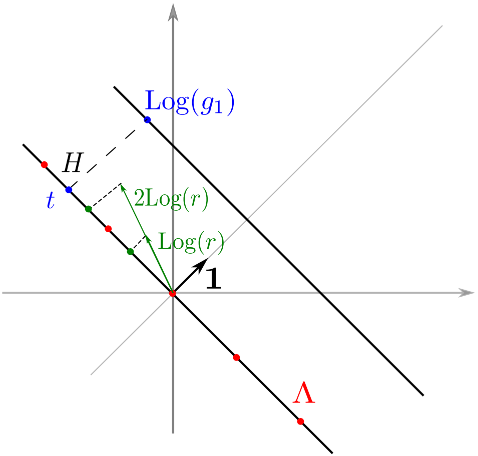

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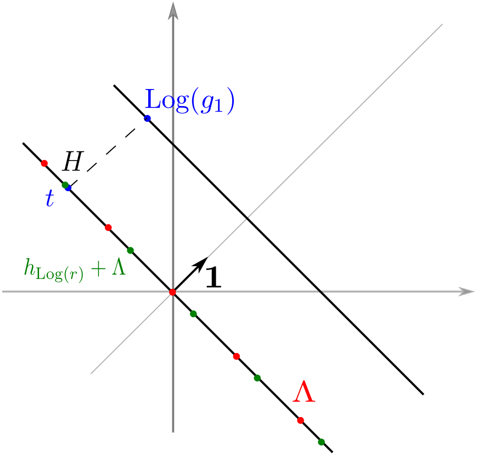

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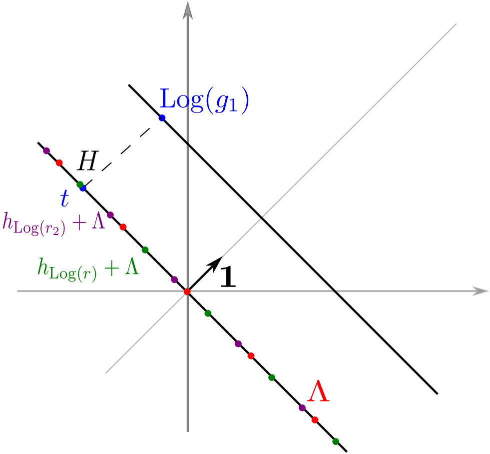

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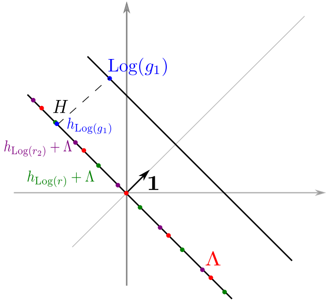
$\mathbf{1}$
$\Lambda$

# Formalisation

**Difficulties**

- We cannot subtract $\mathrm{Log}(r_i)$
- We cannot add too many $\mathrm{Log}(r_i)$'s

$\Rightarrow$ This is not a lattice



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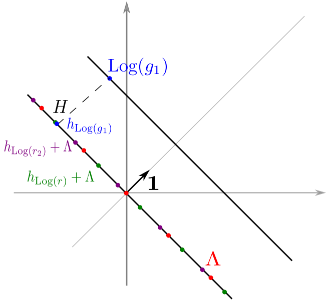


We consider the lattice

$$\begin{array}{|c|cccc|}\hline \Lambda & \multicolumn{4}{c|}{h_{\mathrm{Log}\, r_1}, \dots, h_{\mathrm{Log}\, r_n}} \\ \hline & 1 & & & \\ 0 & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \hline \end{array}$$

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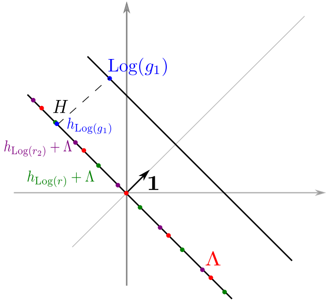


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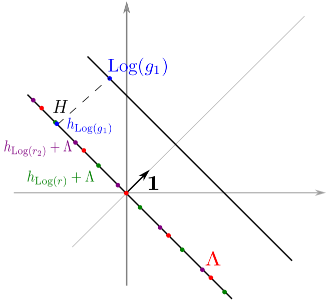


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# How to solve CVP in $L$?

| CDPR | This work |
|---|---|
| Good basis of $\Lambda$ | No good basis of $L$ known |

[Laa16] T. Laarhoven. Finding closest lattice vectors using approximate Voronoi cells. SAC.

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## Key observation

$$L := \left[\begin{array}{c|cccc} \Lambda & h_{\mathrm{Log}\, r_1}, & \dots, & & h_{\mathrm{Log}\, r_n} \\ \hline & 1 & & & \\ 0 & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{array}\right]$$ does not depend on $\langle g \rangle$

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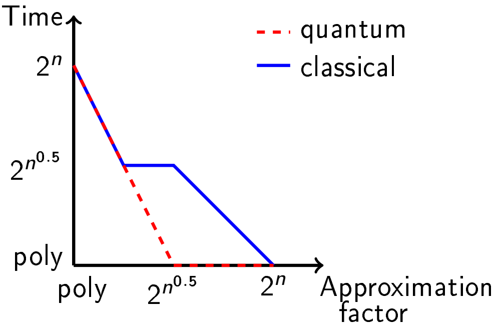
[Laa16]:
- Find $s \in L$ such that $\|s - t\| = \widetilde{O}(n^{\alpha})$
- Time: $2^{\widetilde{O}(n^{1-2\alpha})}$ (query) $+ 2^{O(n)}$ (pre-processing)

[Laa16] T. Laarhoven. Finding closest lattice vectors using approximate Voronoi cells. SAC.

# Conclusion

| Approximation | Query time | Pre-processing |
|---|---|---|
| $2^{\widetilde{O}(n^{\alpha})}$ | $2^{\widetilde{O}(n^{1-2\alpha})} + (\mathrm{poly}(n)$ or $2^{\widetilde{O}(\sqrt{n})})$ | $2^{O(n)}$ |




$+2^{O(n)}$ Pre-processing / Non-uniform algorithm

# Open problems

- Generalization to other number fields?
- Removing (or testing) the heuristics

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Questions?