Introduction to lattice-based cryptography

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Cryptography, Network Security and Cybersecurity webminar, Session-VI

Lattice-based cryptography

lattices, ideal lattices, SVP, CVP, ...	(Ring) LWE (Ring) SIS, NTRU, ...	Regev encryption scheme signatures, trapdoors FHE, obfuscation, functional encryption, ...

+ maths
- crypto

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Outline of the talk

1. Lattices and lattice problems
2. Algorithmic problems for cryptography
3. Cryptographic primitives

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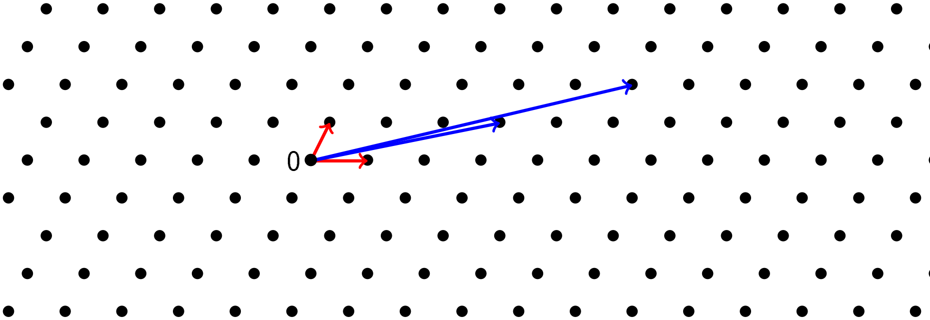
Lattices

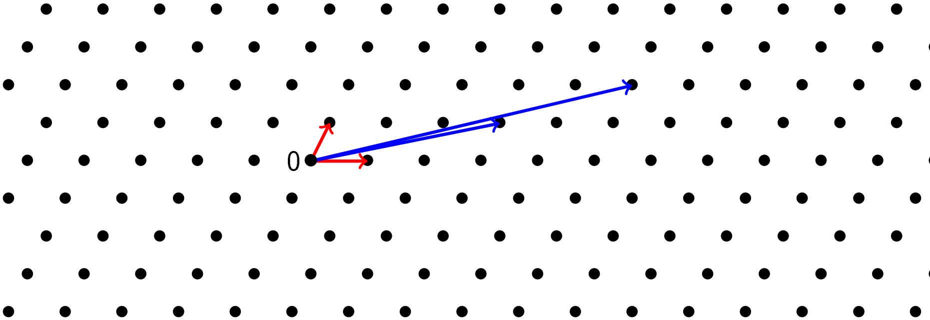


Lattice

A lattice L is a subset of $\mathbb{R}^n$ of the form $L = \{Bx \mid x \in \mathbb{Z}^n\}$, with $B \in \mathbb{R}^{n \times n}$ invertible. B is a basis of L, and n is its rank.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 11 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

Lattices



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We represent a lattice by any of its basis

Algorithmic problems on lattices

Input: any basis of any lattice

Example of problems:

- Testing equality of lattices
- Testing inclusion of lattices
- Intersecting two lattices
- Computing a short vector of a lattice
- Computing a lattice vector close to a target

Algorithmic problems on lattices

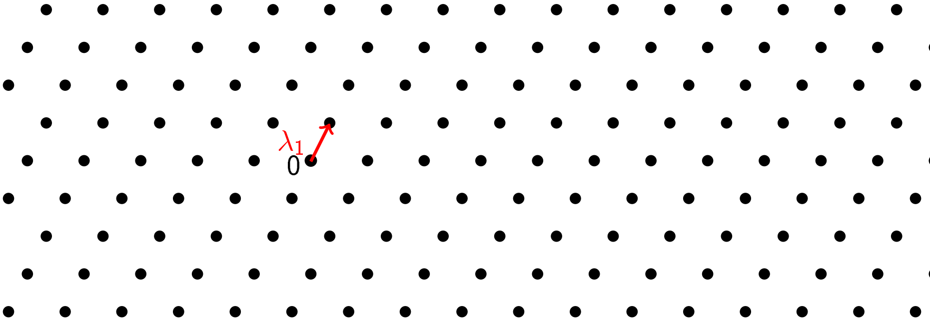
Input: any basis of any lattice

Example of problems:

- Testing equality of lattices $\Rightarrow$ easy
- Testing inclusion of lattices $\Rightarrow$ easy
- Intersecting two lattices $\Rightarrow$ easy
- Computing a short vector of a lattice $\Rightarrow$ hard
- Computing a lattice vector close to a target $\Rightarrow$ hard

easy: polynomial time   hard: no polynomial time algorithm known

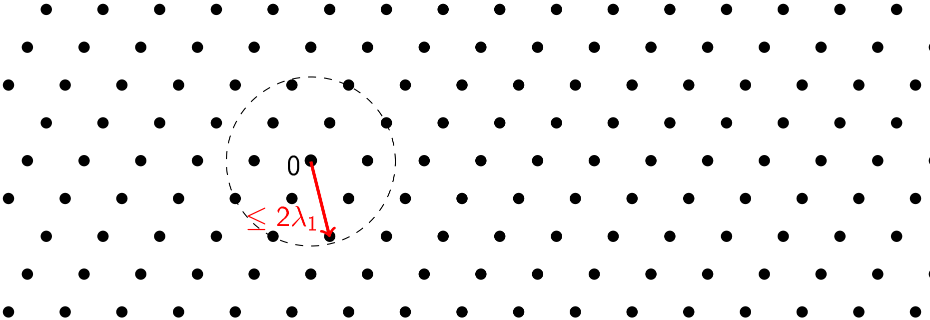
Shortest and Closest vector problems



Shortest Vector Problem (SVP)

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted λ_1.

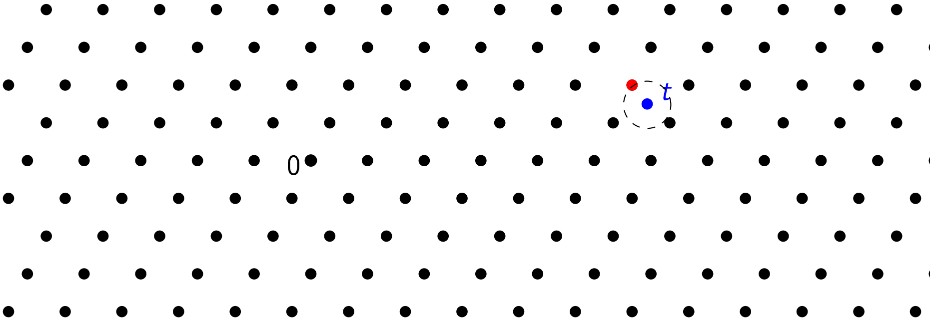
Shortest and Closest vector problems



Approximate Shortest Vector Problem (approx-SVP)

Find a short (in Euclidean norm) non-zero vector.
(e.g. of norm $\leq 2\lambda_1$).

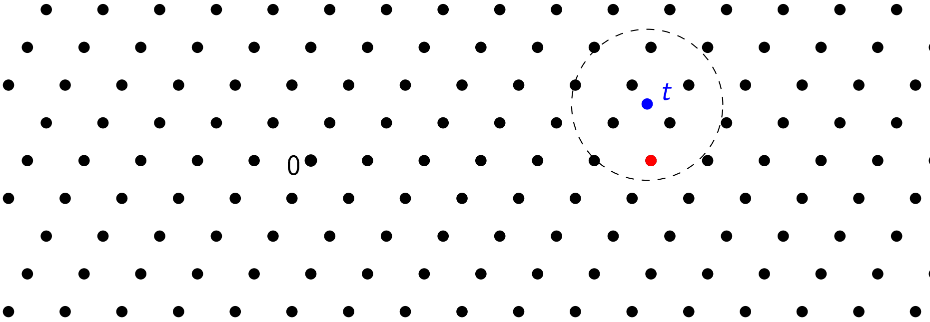
Shortest and Closest vector problems



Closest Vector Problem (CVP)

Given a target point t, find a point of the lattice closest to t.

Shortest and Closest vector problems



Approximate Closest Vector Problem (approx-CVP)

Given a target point t, find a point of the lattice close to t.

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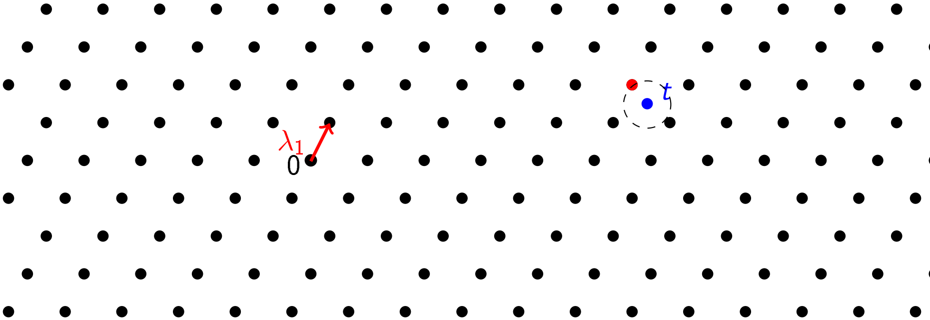
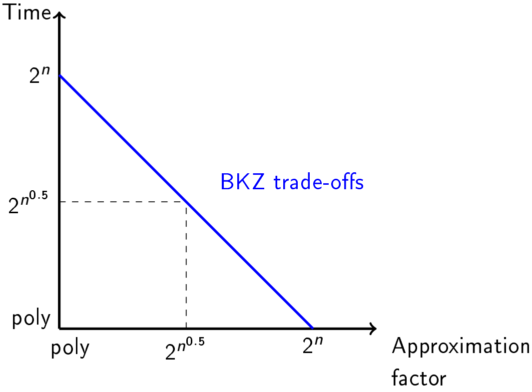


SVP and CVP are hard to solve when n increases

- even with a quantum computer
- even if we allow small approximation factor $(\gamma = \text{poly}(n))$

Hardness of SVP and CVP

Best Time/Approximation trade-off for SVP, CVP (even quantumly):
BKZ algorithm [Sch87,SE94]



[Sch87] C.-P. Schnorr. A hierarchy of polynomial time lattice basis reduction algorithms. TCS.

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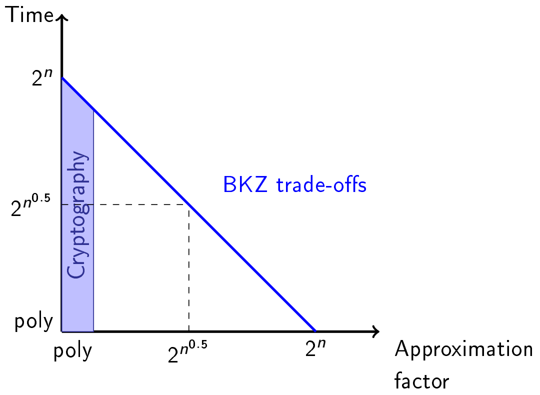

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- up to $n = 80$ or $n = 100 \rightsquigarrow$ a few minutes on a personal laptop
- up to $n = 170 \rightsquigarrow$ a few days on a big computer with optimized code
- from $n = 500$ to $n = 1000 \rightsquigarrow$ cryptography

An example of lattice reduction algorithm

The Lagrange-Gauss algorithm:

- For lattices of rank $n = 2$ only
- Solves exact SVP
- Polynomial time

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video

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But remember: when n is large, solving exact SVP is hard

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Limitations of SVP (and CVP)

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For crypto, we need problems that are hard on average

(i.e., for a random instance, the problem is hard with overwhelming probability)

The SIS problem

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

SIS (Short Integer Solution) [Ajt96]

Given $A \leftarrow \mathrm{Uniform}(\mathbb{Z}_q^{m \times n})$ (with $n \log q < m$)

Find $x \in \{-B, \cdots, B\}^m \setminus \{0\}$ s.t. $x \, A = 0 \bmod q$

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Solving SIS with non-negligible probability (e.g., $\geq 2^{-80}$) $\overset{\sim}{\iff}$ Solving SVP in any lattice of rank n

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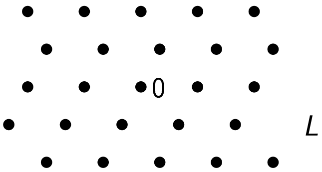

$$L = \{x \in \mathbb{Z}^m \,|\, xA = 0 \bmod q\}$$

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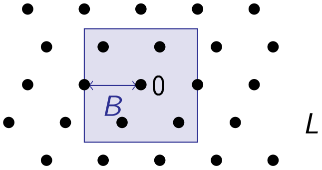

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SIS ≈ SVP in L

The LWE problem

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LWE (Learning With Errors) [Reg05]

Sample $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{n \times n})$ and s, $e \leftarrow \text{Uniform}(\{-B, \cdots, B\}^n)$

Given A and b, where $b := A s + e \bmod q$

Recover s or e

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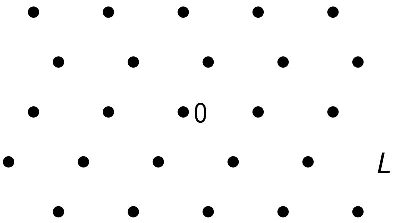

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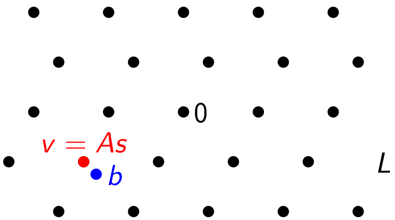

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$b = v + e$,
where $v \in L$ and e small

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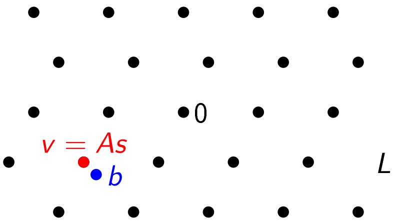

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LWE $\approx$ CVP in L

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SIS $\overset{\sim}{\longleftrightarrow}$ average case SVP

LWE $\overset{\sim}{\longleftrightarrow}$ average case CVP

Decision variant of LWE

decision-LWE

Sample $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{n\times n})$ and s, $e \leftarrow \text{Uniform}(\{-B, \cdots, B\}^n)$

Given A and b, where

$$b := A s + e \bmod q \quad \text{or} \quad b \leftarrow \text{Uniform}(\mathbb{Z}_q^n)$$

Guess whether b is uniform or not.

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$\Rightarrow$ decision problems can be easier to use for crypto

Outline of the talk

Collision-resistant hash functions

$\mathcal{G} = \{H : S \to S'\}$ is a family of collision-resistant hash functions if

- it is compressing: $|S'| < |S|$
- it is collision-resistant: $\forall$ PPT adversary $\mathcal{A}$,

$$\Pr_{H \leftarrow \mathrm{Uniform}(\mathcal{G})}\Big[(x_1, x_2) \leftarrow \mathcal{A}(H) \,|\, x_1 \neq x_2,\ H(x_1) = H(x_2)\Big] \leq \mathrm{negl}$$

SIS-based collision-resistant hash functions

$\mathcal{G} = \{H_A \,|\, A \in \mathbb{Z}_q^{m \times n}\}$, where

$$H_A : \{0,1\}^m \to \{0, \cdots, q-1\}^n$$
$$x \mapsto x \, A \mod q$$

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- collision-resistance: $\mathcal{A}$ breaking $\mathcal{G} \Rightarrow \mathcal{A}_{SIS}$ breaking SIS

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  $\mathcal{A}_{SIS}(A)$:
  - $(x_1, x_2) \leftarrow \mathcal{A}(H_A)$ $(x_1 A = x_2 A \bmod q)$
  - output $x_1 - x_2$ $(\in \{-B, \cdots, B\}^m$ since $B \geq 1)$

Encryption scheme

$\mathsf{KeyGen}(1^\lambda) = (\mathrm{sk}, \mathrm{pk})$
$\mathsf{Enc}(\mathrm{pk}, m \in \{0, 1\}) = c$
$\mathsf{Dec}(\mathrm{sk}, c) = \overline{m}$

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- Correction: $\forall (sk, pk) \leftarrow \mathsf{KeyGen}(1^\lambda)$, $\forall m \in \{0,1\}$,

$$\mathsf{Dec}(\mathrm{sk}, \mathsf{Enc}(\mathrm{pk}, m)) = m$$

- CPA security: $\forall$ PPT adversary $\mathcal{A}$,

$$\Big| \Pr_{(sk,pk)\leftarrow \mathsf{KeyGen}(1^\lambda)} \Big[\mathcal{A}(\mathrm{pk}, c) = 1 \mid c \leftarrow \mathsf{Enc}(\mathrm{pk}, 0)\Big] - \Pr_{(sk,pk)\leftarrow \mathsf{KeyGen}(1^\lambda)} \Big[\mathcal{A}(\mathrm{pk}, c) = 1 \mid c \leftarrow \mathsf{Enc}(\mathrm{pk}, 1)\Big]\Big| = \mathrm{negl}$$

LWE-based encryption (Regev's Encryption)

KeyGen(1^λ):

- sample $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{n \times n})$
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Enc(pk, m):

- sample $s' \leftarrow \text{Uniform}(\{-B,\cdots,B\}^n)$
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Security: $b \approx b \leftarrow \text{Uniform}(\mathbb{Z}_q^n)$ (by decision-LWE)

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(by transposing decision-LWE)

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Security: $(b' + \lfloor \frac{q}{2} \rceil \, 0...0m \mod q)$ uniform in $\mathbb{Z}_q^{n+1}$

$\Rightarrow$ independent of m

Decryption and correction

Reminder

$c = \boxed{s'} \cdot \boxed{A \mid b} + \boxed{e'} + \lfloor \frac{q}{2} \rceil \boxed{0 \dots 0 m} \bmod q$ and $\mathrm{sk} = \boxed{s}$

Dec(sk, c):

- $x = \boxed{c} \cdot \begin{pmatrix} s \\ -1 \end{pmatrix} \bmod q$ $(x \in [-q/2, q/2])$
- if $|x| < q/4$ output 0
- otherwise output 1

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Correction:

$$c \cdot \begin{pmatrix} s \\ -1 \end{pmatrix} = s' A s - s' b + e' \begin{pmatrix} s \\ -1 \end{pmatrix} - m \cdot \lfloor \frac{q}{2} \rceil$$

Decryption and correction

Reminder

$c = \boxed{s'} \cdot \boxed{A \mid b} + \boxed{e'} + \lfloor \frac{q}{2} \rceil \boxed{0...0m} \mod q$ and $\mathrm{sk} = \boxed{s}$

Dec(sk, c):
- $x = \boxed{c} \cdot \begin{bmatrix} s \\ -1 \end{bmatrix} \mod q \quad (x \in [-q/2, q/2])$
- if $|x| < q/4$ output 0
- otherwise output 1

Correction:

$$\boxed{c} \cdot \begin{bmatrix} s \\ -1 \end{bmatrix} = \boxed{s'} \boxed{A} \boxed{s} - \boxed{s'} (\boxed{A} \boxed{s} + \boxed{e}) + \boxed{e'} \begin{bmatrix} s \\ -1 \end{bmatrix} - m \cdot \lfloor \frac{q}{2} \rceil$$

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$$= -\boxed{s'}\,\boxed{e} + \boxed{e'} \begin{pmatrix} s \\ \text{-1} \end{pmatrix} - m \cdot \lfloor \frac{q}{2} \rceil$$

Decryption and correction

Reminder

$c = s' \cdot \begin{pmatrix} A & b \end{pmatrix} + e' + \lfloor \frac{q}{2} \rceil \begin{pmatrix} 0 \ldots 0 m \end{pmatrix} \bmod q$ and $\mathrm{sk} = s$

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- $x = c \cdot \begin{pmatrix} s \\ -1 \end{pmatrix} \bmod q$ $(x \in [-q/2, q/2])$
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Correction:

$$c \cdot \begin{pmatrix} s \\ -1 \end{pmatrix} = s' A s - s'(A s + e) + e' \begin{pmatrix} s \\ -1 \end{pmatrix} - m \cdot \lfloor \frac{q}{2} \rceil$$

$$= \quad \text{small} \quad - m \cdot \lfloor \frac{q}{2} \rceil$$

NIST post-quantum standardization process

Objective: new standard for post-quantum encryption (and signature)

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- Started in 2017 $\rightsquigarrow$ 48 encryption candidates
- Since August 2020 (round 3) $\rightsquigarrow$ 4 candidates left
    - 3 of them are based on lattices

Conclusion

Structured lattices

Reminder

Lattices are represented by a basis B.

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Structured basis: $B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ b_n & b_1 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_2 & b_3 & \cdots & b_1 \end{pmatrix}$ $\rightsquigarrow$ n storage (e.g., RLWE)

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- schemes more efficient
- are they still secure?

Take-away

- Wide range of possible questions related to lattice-based crypto

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Questions?