# Approx-SVP in Ideal lattices with Pre-Processing

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LIP, ENS de Lyon

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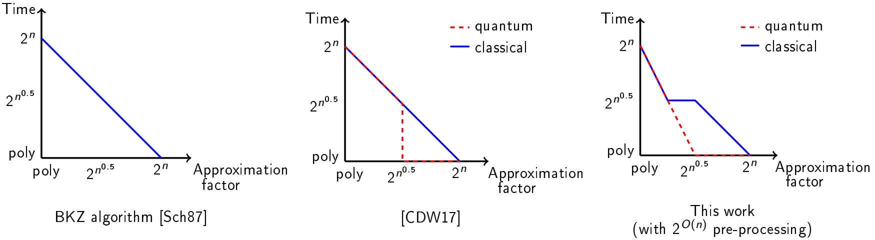
May 24, 2019

https://eprint.iacr.org/2019/215.pdf

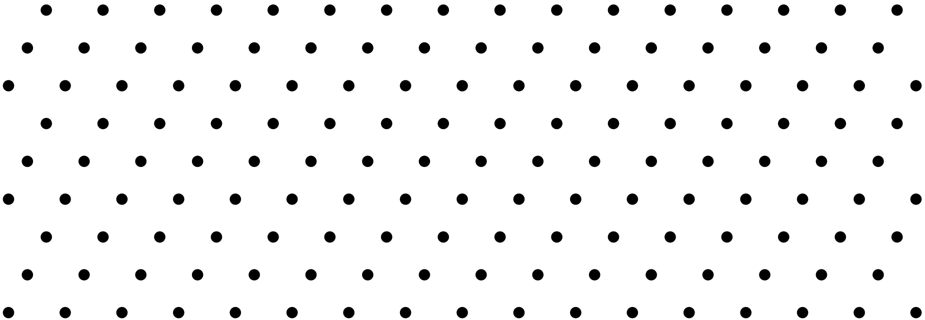
# What is this talk about

Time/Approximation trade-offs for SVP in ideal lattices:




(Figures are for prime power cyclotomic fields)

# Lattices



## Lattice

A lattice $L$ is a discrete 'vector space' over $\mathbb{Z}$.

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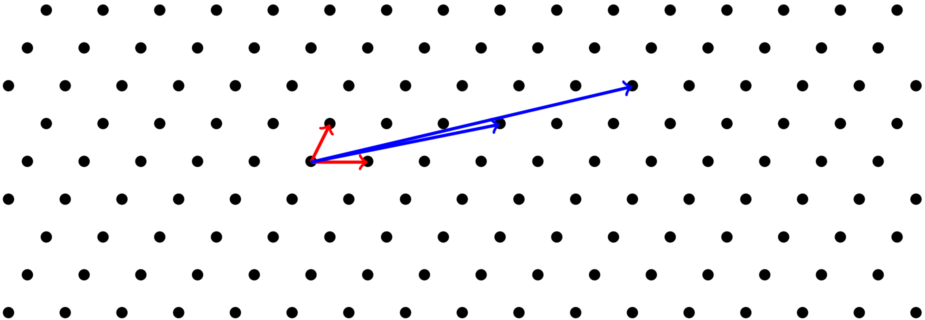
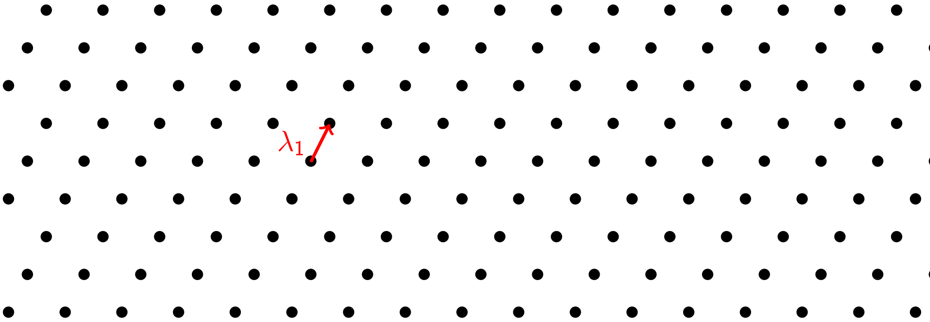


## Lattice

A lattice $L$ is a discrete 'vector space' over $\mathbb{Z}$.
A basis of $L$ is an invertible matrix $B$ such that $L = \{Bx \mid x \in \mathbb{Z}^n\}$.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 11 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

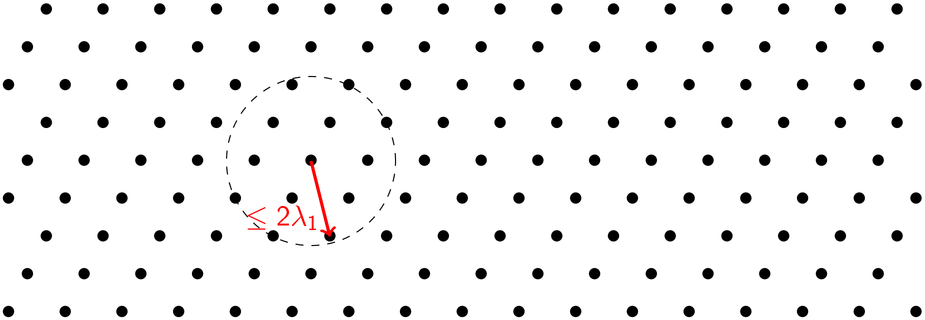
# Lattices




**Shortest Vector Problem (SVP)**

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted $\lambda_1$.

# Lattices




**Approximate Shortest Vector Problem (approx-SVP)**

Find a short (in Euclidean norm) non-zero vector.
(e.g. of norm $\leq 2\lambda_1$).

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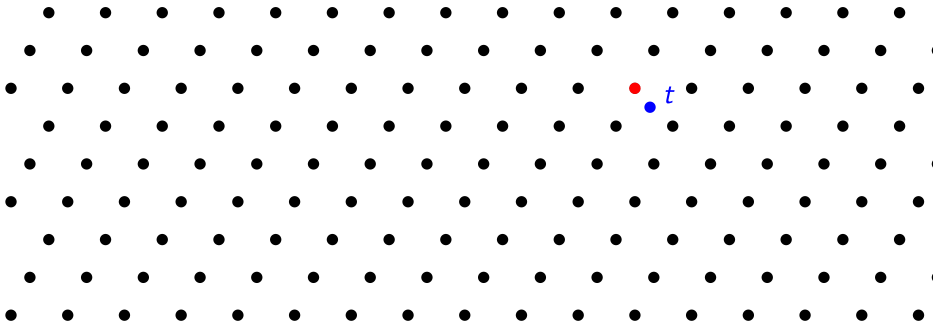


## Closest Vector Problem (CVP)

Given a target point $t$, find a point of the lattice closest to $t$.

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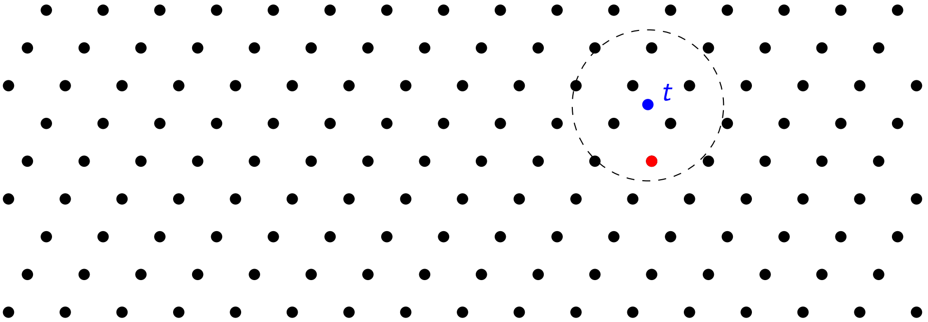


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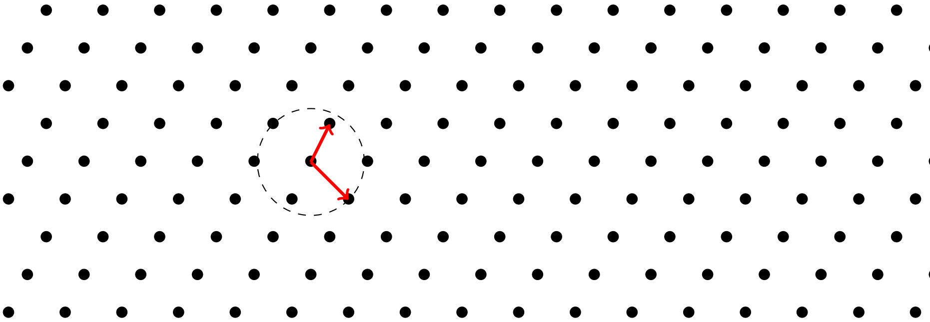

## Shortest Independent Vectors Problem (SIVP)

Find $n$ short vectors which are linearly independent.

# Complexity of SVP/CVP

**Applications**

SVP and CVP in general lattices are conjectured to be hard to solve both quantumly and classically $\Rightarrow$ used in cryptography

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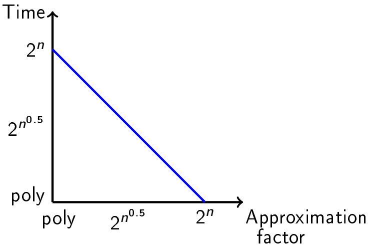

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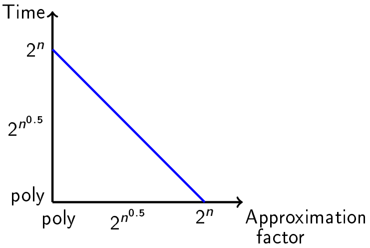


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$\boxed{\text{LWE / SIS}} \longrightarrow \boxed{\text{SIVP}}$

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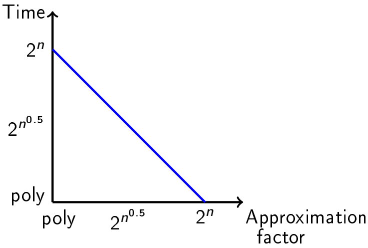


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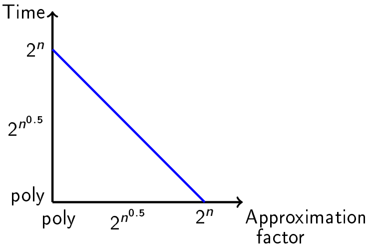


**Applications**

$\boxed{\text{LWE / SIS}} \rightleftarrows \boxed{\text{SIVP}}$

- hash functions
- encryption
- identity based encryption
- fully homomorphic encryption
- ...

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Improve efficiency using structured lattices, e.g.

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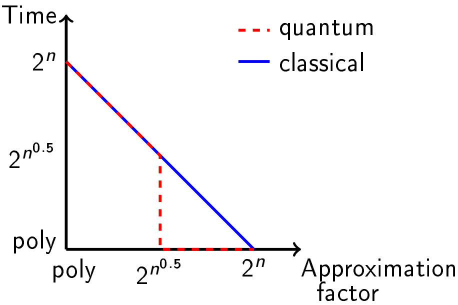
*Is approx-SVP still hard when restricted to ideal lattices?*

# SVP in ideal lattices

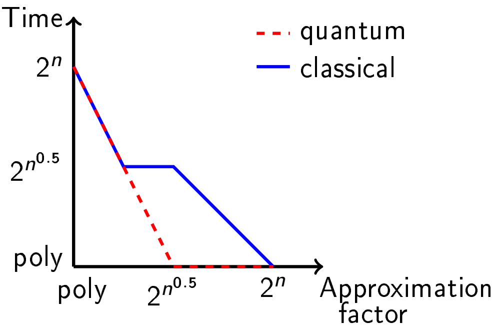
[CDW17]: Better than BKZ in the quantum setting




- Heuristic
- For prime power cyclotomic fields

[CDW17] R. Cramer, L. Ducas, B. Wesolowski. Short Stickelberger Class Relations and Application to Ideal-SVP, Eurocrypt.

# This work




(Figure for prime power cyclotomic fields)

- Heuristic
- Pre-processing $2^{O(n)}$, independent of the choice of the ideal
- All number fields (trade-offs differ slightly)

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[Gen09] C. Gentry. Fully homomorphic encryption using ideal lattices, STOC.

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$$\begin{tikzcd}
\boxed{\text{Schemes}} \arrow[r] & \boxed{\text{RLWE}} \arrow[r, shift left] \arrow[d, leftrightarrow] & \boxed{\text{ideal SVP}} \arrow[l, shift left, "?"] \\
& \boxed{\text{module SIVP}} &
\end{tikzcd}$$

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# Outline of the talk

# First definitions

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- Units: $R^\times = \{a \in R \mid \exists b \in R, ab = 1\}$
  - e.g. $\mathbb{Z}^\times = \{-1, 1\}$
- Principal ideals: $\langle g \rangle = \{gr \mid r \in R\}$ (i.e. all multiples of $g$)
  - e.g. $\langle 2 \rangle = \{\text{even numbers}\}$ in $\mathbb{Z}$
  - $g$ is called a generator of $\langle g \rangle$
  - the generators of $\langle g \rangle$ are exactly the $ug$ for $u \in R^\times$

# Why is $\langle g \rangle$ a lattice?

**$R \simeq \mathbb{Z}^n$**

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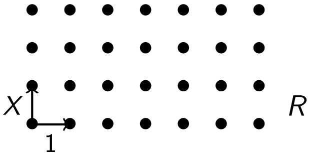

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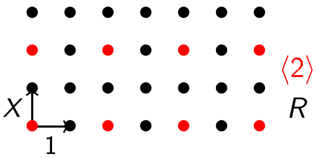

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Given a basis of a principal ideal $\langle g \rangle$ and $\alpha \in (0, 1]$,
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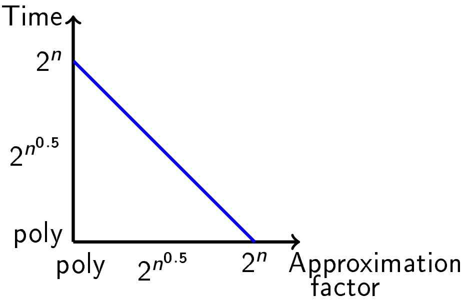
BKZ algorithm can do it in time $2^{O(n^{1-\alpha})}$, can we do better?



# Outline of the talk

# Main idea of the CDPR algorithm (on an idea of [CGS14])

**Idea**

Maybe $g$ is a somehow small element of $\langle g \rangle$

[CGS14]: P. Campbell, M. Groves, and D. Shepherd. Soliloquy: A cautionary tale.

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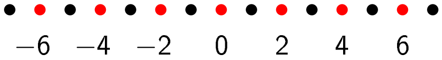


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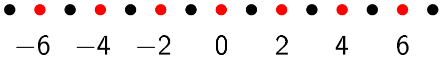


**For larger n:** one of the generators is somehow small

---

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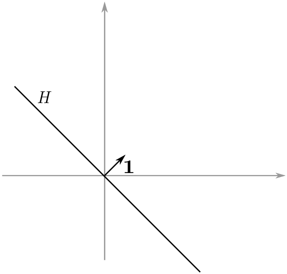
Log : $R \rightarrow \mathbb{R}^n$ (somehow generalising log to $R$)

Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^{\perp}$.



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$\text{Log}\, r = h + a\mathbf{1}$, with $h \in H$

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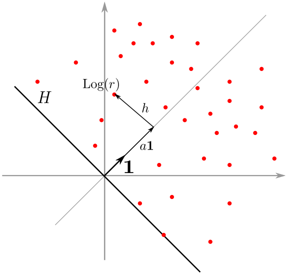

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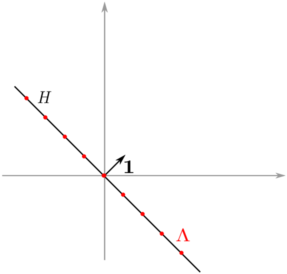

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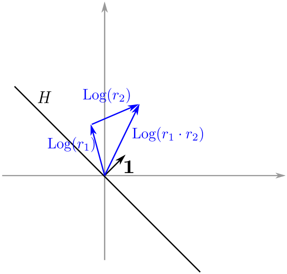

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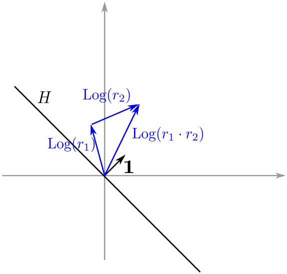
- $\|r\| \simeq 2^{\|\text{Log}\, r\|_\infty}$



# The CDPR algorithm

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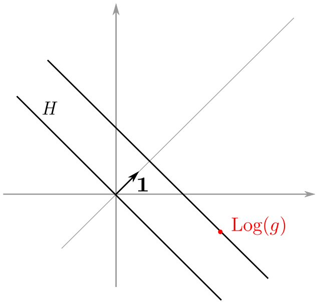

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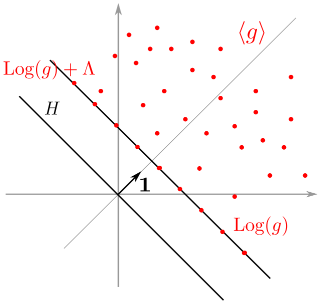

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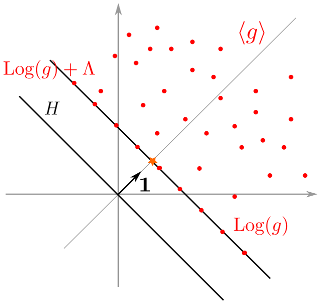

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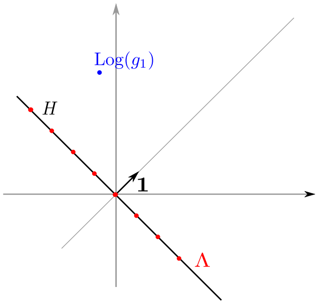


[BS16]: J.F. Biasse, F. Song. Efficient quantum algorithms for computing class groups and solving the principal ideal problem in arbitrary degree number fields, SODA.

[BEFGK17]: J.F. Biasse, T. Espitau, P.A. Fouque, A. Gélin, P. Kirchner. Computing generator in cyclotomic integer rings, Eurocrypt.

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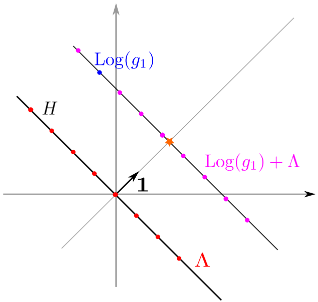


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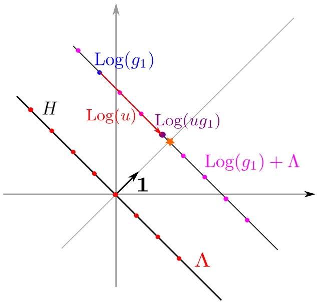


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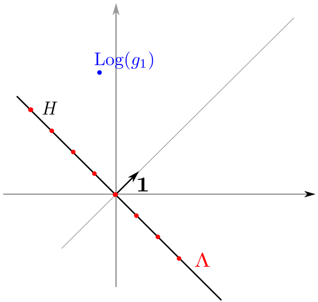


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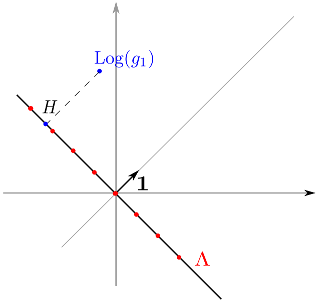


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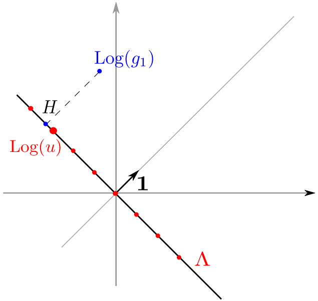


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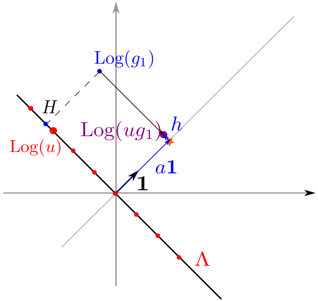


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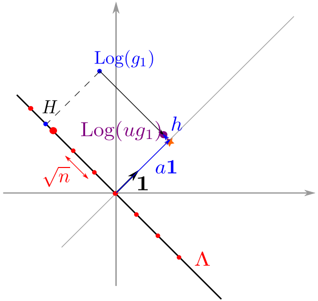


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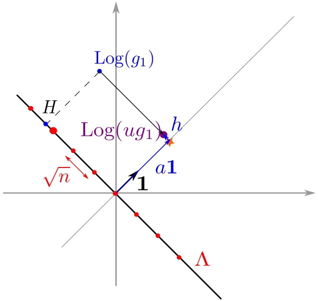


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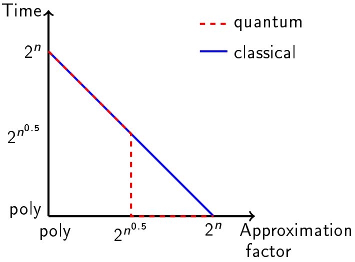


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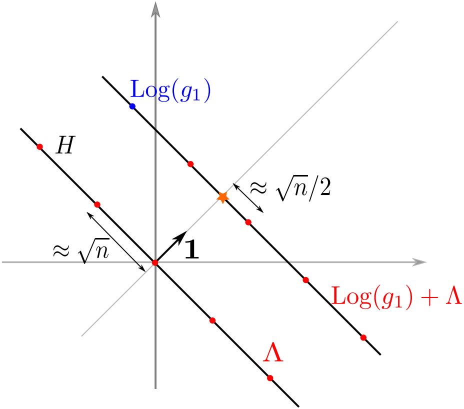
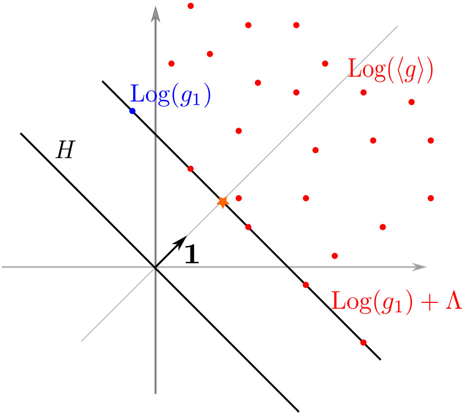
# Outline of the talk

# Idea



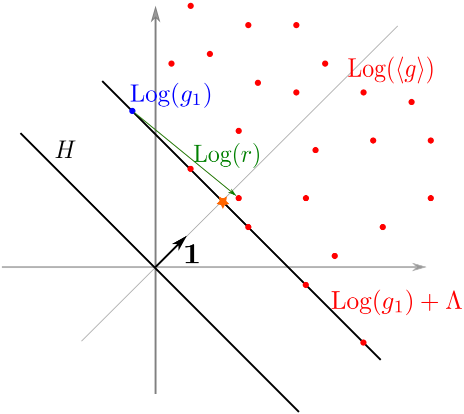
$\mathrm{Log}(g_1)$
$H$
$\approx \sqrt{n}/2$
$\approx \sqrt{n}$
$\mathbf{1}$
$\mathrm{Log}(g_1) + \Lambda$
$\Lambda$

# Idea



$\mathrm{Log}(\langle g \rangle)$
$\mathrm{Log}(g_1)$
$H$
$\mathbf{1}$
$\mathrm{Log}(g_1) + \Lambda$

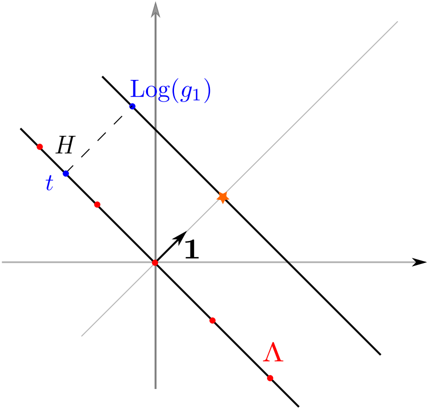
# Idea




## Important

$\mathrm{Log}\, r = h + a\mathbf{1}$ with $a$ small (and $h \in H$).

# Idea

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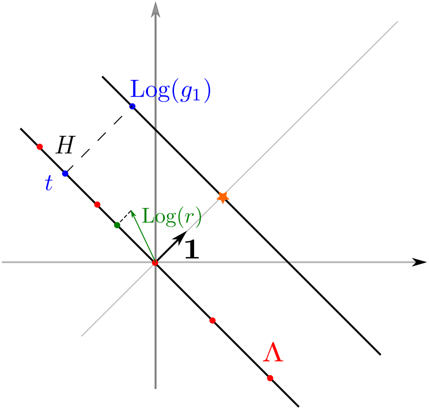


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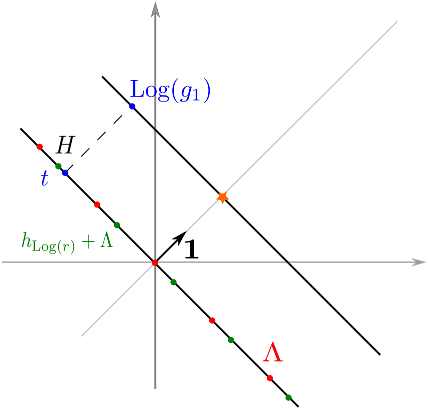


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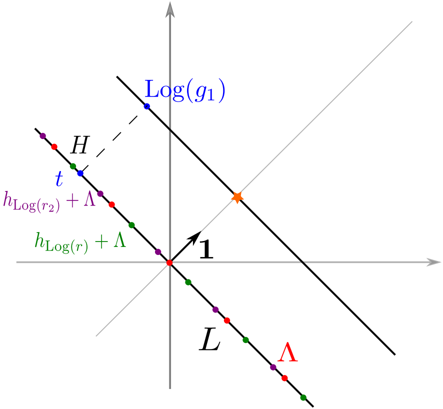


## Important

$\operatorname{Log} r = h + a\mathbf{1}$ with $a$ small (and $h \in H$).

# The lattice $L$

$$
L = \left(\begin{array}{c|cccc}
\Lambda & h_{\mathrm{Log}(r_1)}, & \cdots & , h_{\mathrm{Log}(r_\nu)} & \\
\hline
 & 1 & & & \\
0 & & 1 & & \\
 & & & \ddots & \\
 & & & & 1
\end{array}\right)
\qquad
t = \left(\begin{array}{c}
h_{Log(g_1)} \\
\hline
\\
0 \\
\\
\end{array}\right)
$$

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## Heuristic

For some $\nu = \widetilde{O}(n)$, the covering radius of $L$ satisfies $\mu(L) = O(1)$.

(= for all target $t$, there exists $s \in L$ such that $\|t - s\| = O(1)$)

# Algorithm

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Compute $r_1, \cdots, r_n$ with small '$a$'

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$\Rightarrow$ get a vector $s \in L$ such that $\|s - t\| \leq \widetilde{O}(n^\alpha)$

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$$\boxed{\|rg_1\| \leq 2^{\widetilde{O}(n^\alpha)} \cdot \lambda_1}$$

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Compute $r_1, \cdots, r_n$ with small '$a$' $\quad$ $\text{poly}(n)$ / $2^{\widetilde{O}(\sqrt{n})}$

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# Algorithm

Compute $r_1, \cdots, r_n$ with small '$a$' $\quad \mathrm{poly}(n)$ / $2^{\widetilde{O}(\sqrt{n})}$

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Compute the lattice $L$ $\quad \mathrm{poly}(n)$

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Solve CVP in $L$ with target $t$ (for some $\alpha \in [0,1]$) $\quad$ ?
$\Rightarrow$ get a vector $s \in L$ such that $\|s - t\| \leq \widetilde{O}(n^\alpha)$

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# How to solve CVP in $L$?

| CDPR | This work |
|---|---|
| Good basis of $\Lambda$ | No good basis of $L$ known |

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## Key observation

$L$ does not depend on $\langle g \rangle$

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$L$ does not depend on $\langle g \rangle \quad \Rightarrow$ Pre-processing on $L$

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**Key observation**

$L$ does not depend on $\langle g \rangle$ $\Rightarrow$ Pre-processing on $L$

[Laa16,DLW19,Ste19]:

- Find $s \in L$ such that $\|s - t\| = \widetilde{O}(n^{\alpha})$
- Time:
  - $2^{\widetilde{O}(n^{1-2\alpha})}$ (query)
  - $+\ 2^{O(n)}$ (pre-processing)

---

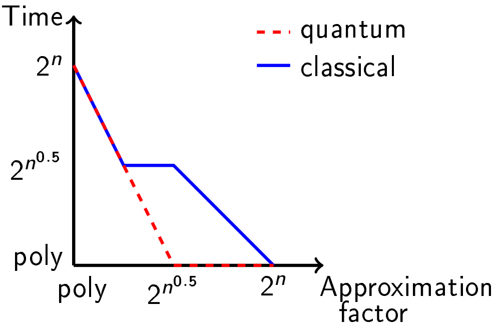
[Laa16] T. Laarhoven. Finding closest lattice vectors using approximate Voronoi cells. SAC.

[DLW19]: E. Doulgerakis, T. Laarhoven, and B. de Weger. Finding closest lattice vectors using approximate Voronoi cells. PQCRYPTO.

[Ste19]: N. Stephens-Davidowitz. A time-distance trade-off for GDD with preprocessing – instantiating the DLW heuristic. ArXiv.

# Conclusion

| Approximation | Query time | Pre-processing |
|---|---|---|
| $2^{\widetilde{O}(n^{\alpha})}$ | $2^{\widetilde{O}(n^{1-2\alpha})} + (\mathrm{poly}(n)$ or $2^{\widetilde{O}(\sqrt{n})})$ | $2^{O(n)}$ |




$+2^{O(n)}$ Pre-processing / Non-uniform algorithm

# Outline of the talk

# Extensions

We can extend the algorithm to

- Non-principal ideals

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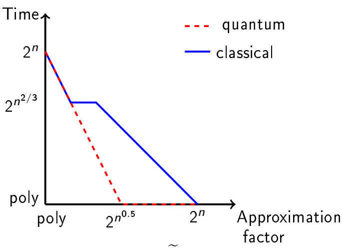
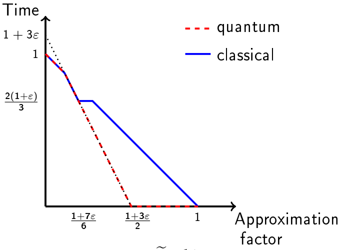
We can extend the algorithm to

- Non-principal ideals
- All number fields




$\log|\Delta| = \widetilde{O}(n)$




$\log|\Delta| = \widetilde{O}(n^{1+\varepsilon})$

# Extension to any ideal

## Ideal

An ideal is $I = \{ar_1 + br_2,\ r_1, r_2 \in R\}$ for some $a, b \in R$
A principal ideal is $\langle g \rangle = \{gr, r \in R\}$ for some $g \in R$.

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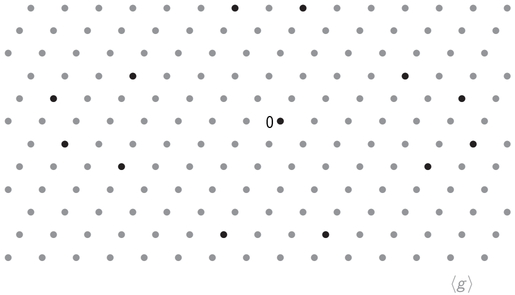

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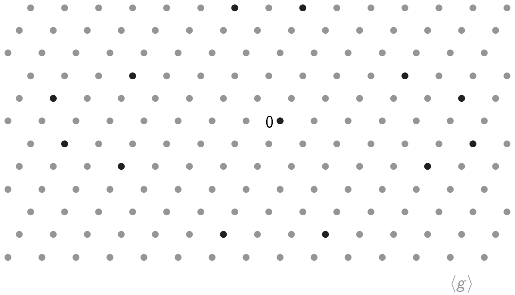


What we did

- All generators • are somehow large

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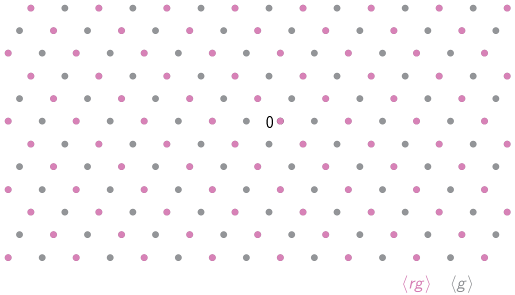


What we did

- All generators • are somehow large
- Multiply by some small $r$
  - $\langle rg \rangle$ sublattice of $\langle g \rangle$

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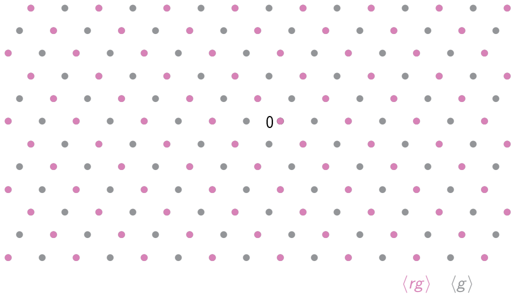


What we did

- All generators • are somehow large
- Multiply by some small $r$
  - $\langle rg \rangle$ sublattice of $\langle g \rangle$
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An ideal is $I = \{ar_1 + br_2,\ r_1, r_2 \in R\}$ for some $a, b \in R$
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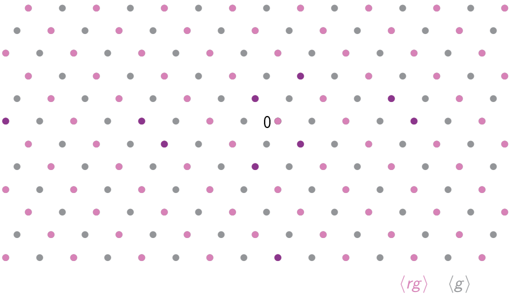


What we did

- All generators • are somehow large
- Multiply by some small $r$
  - $\langle rg \rangle$ sublattice of $\langle g \rangle$
  - not much sparser
  - with a small generator •

# Extension to any ideal

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An ideal is $I = \{ar_1 + br_2,\ r_1, r_2 \in R\}$ for some $a, b \in R$
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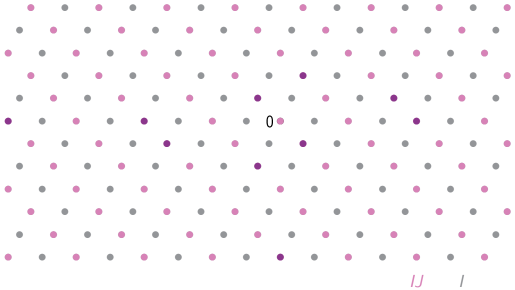


Extension to any ideal

- $I$ has no generator (not principal)
- Multiply by some small ideal $J$
  - $IJ$ sublattice of $I$
  - not much sparser
  - principal
  - with a small generator •

# Perspectives and open questions

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[BBV+17] J. Bauch, D. J. Bernstein, H. de Valence, T. Lange, C. van Vredendaal. Short generators without quantum computers: the case of multiquadratics. Eurocrypt.

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