# Lattice-based cryptography,

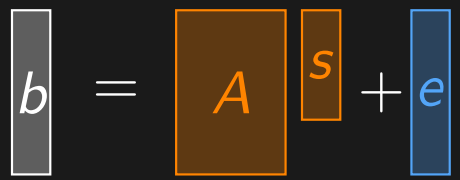
## the picture way

Alice Pellet-Mary

SCN 2024, Amalfi


# Two lattice worlds

## Matrix formalism

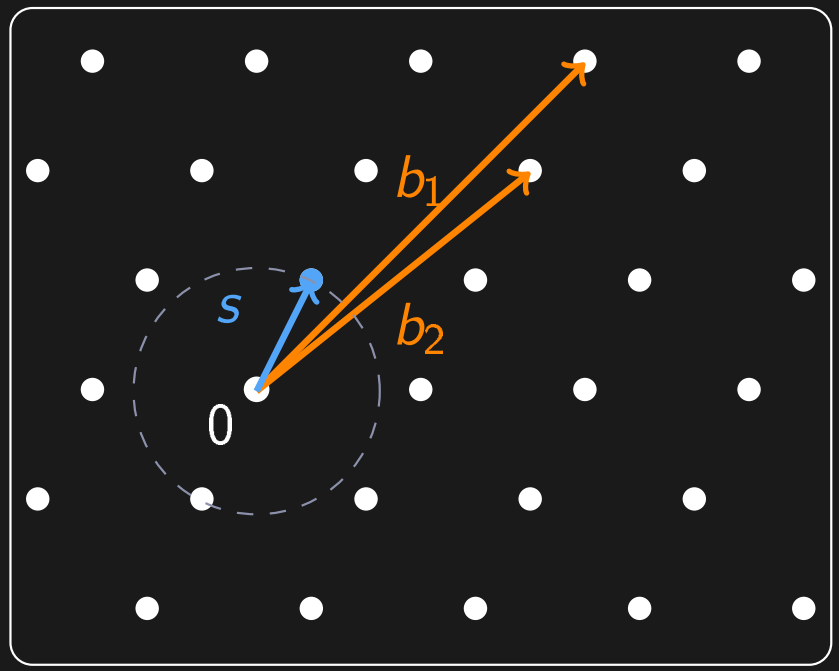



Orange: uniform in $\mathbb{Z}/q\mathbb{Z}$

Blue: uniform in $\{-1, 0, 1\}$

**LWE**: given $A$ and $b$, find $s$

## Geometric formalism




**SVP**: given $(b_1, b_2)$, find $s$

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$$b = A\,s + e$$

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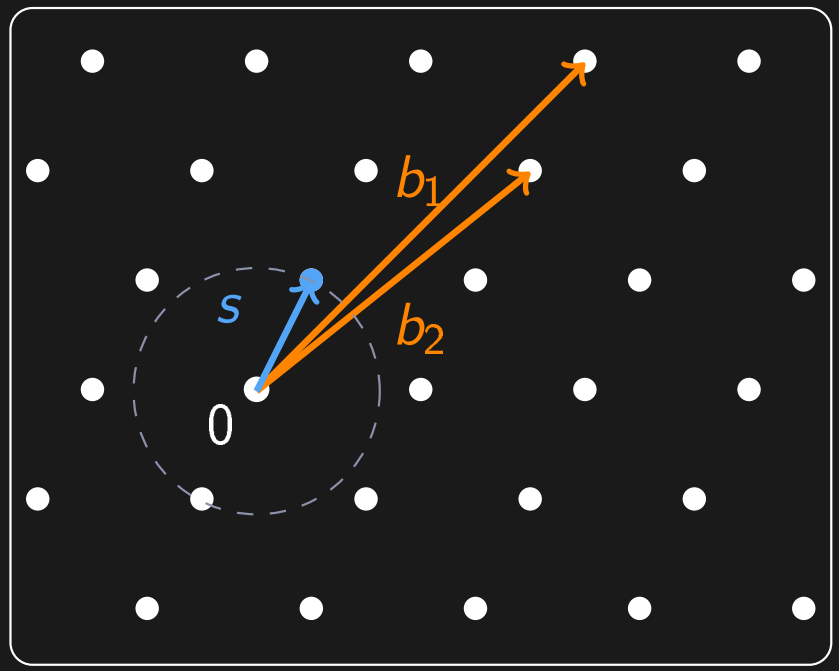


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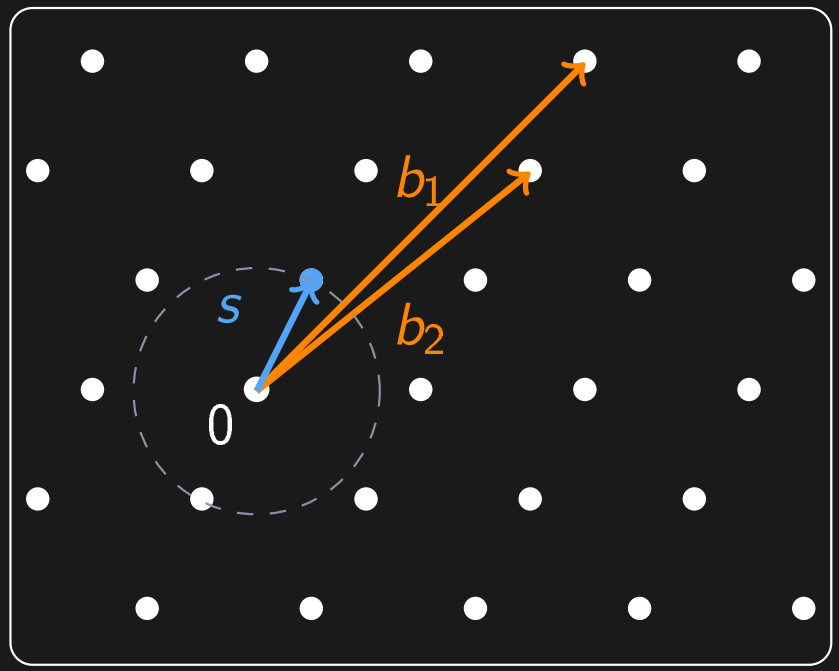


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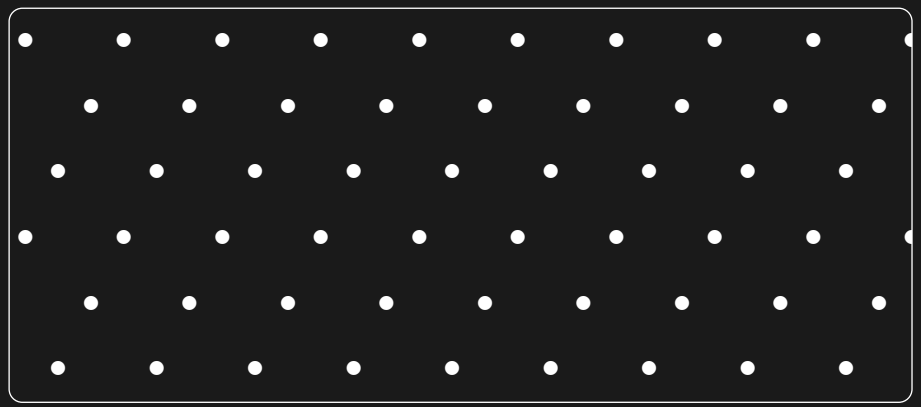
# Plan of the talk

Introduction: Lattices and some definitions

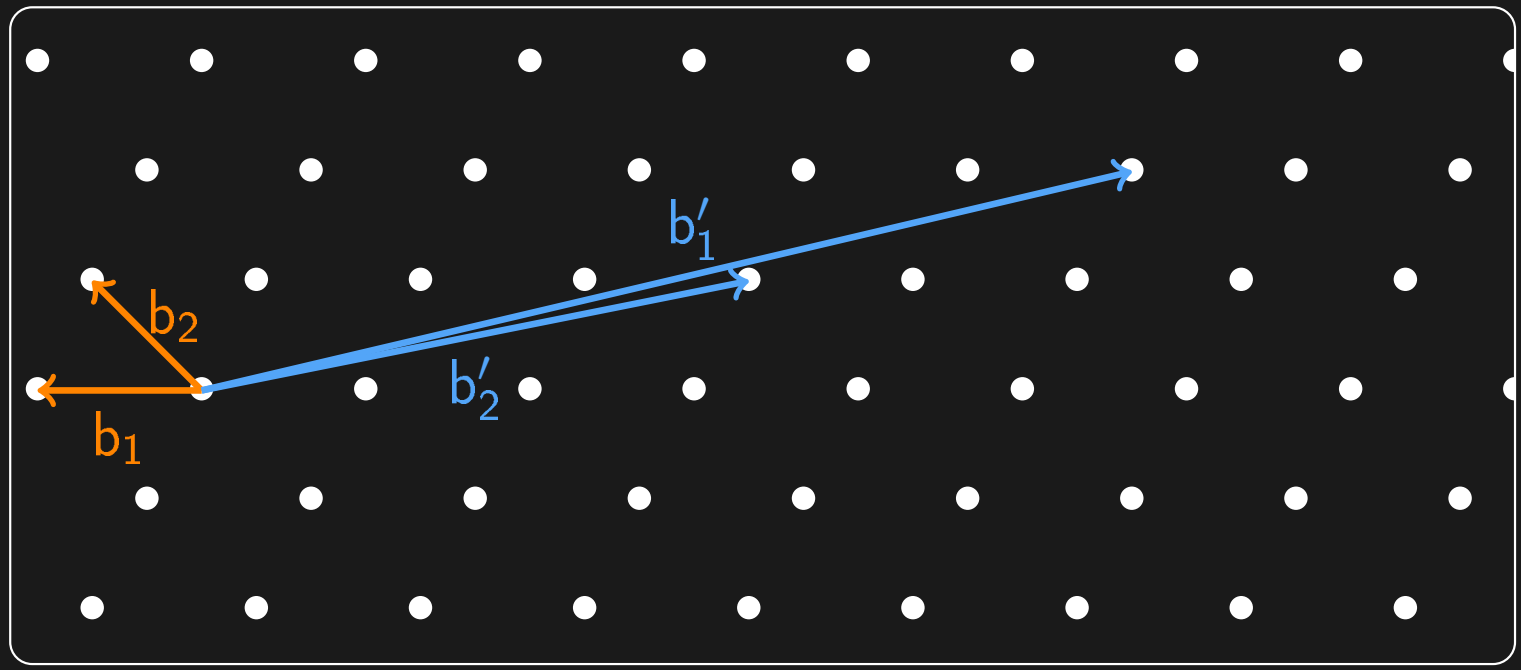
1. Hash-and-sign signatures
2. Attack on a variant of SIS
3. The NTRU problem

} all independent

# Lattices and some definitions



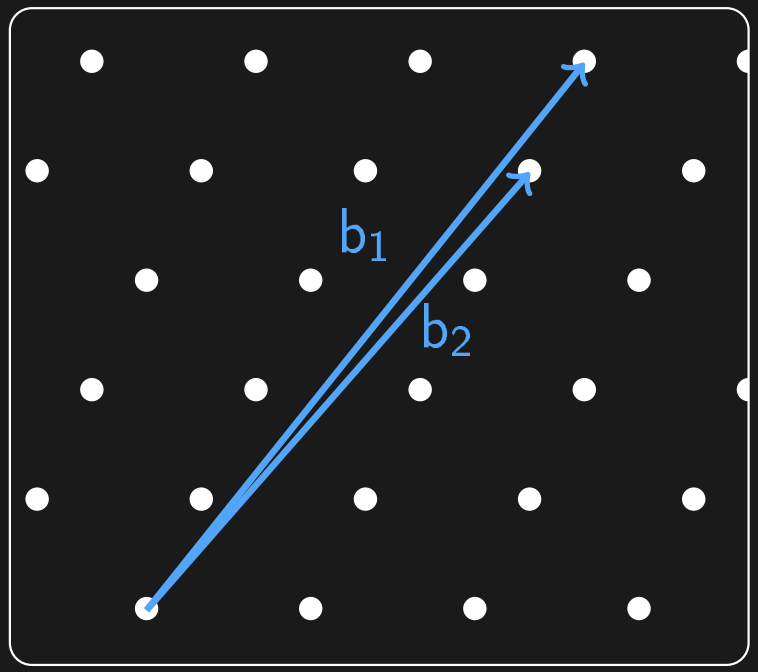
# Lattices




- $\mathcal{L} = \{\sum_{i=1}^{n} x_i b_i \mid \forall i,\, x_i \in \mathbb{Z}\}$ is a lattice
- $(b_1, \ldots, b_n) =: B \in \mathrm{GL}_n(\mathbb{R})$ is a basis (not unique)

# Short basis problem

Input:

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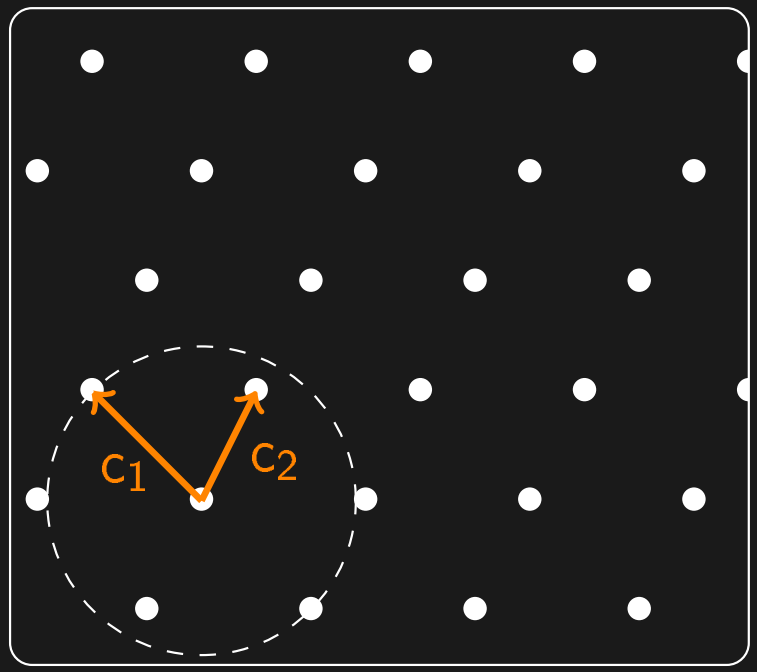


Shortest basis problem

$$\max_i \|c_i\| \leq \min_{B' \text{ basis of } L} \left( \max_i \|b'_i\| \right)$$

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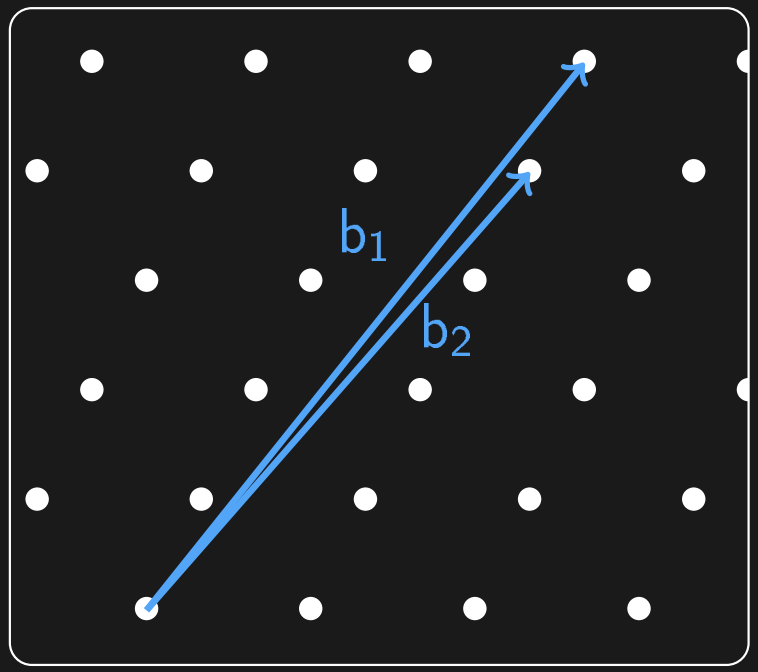


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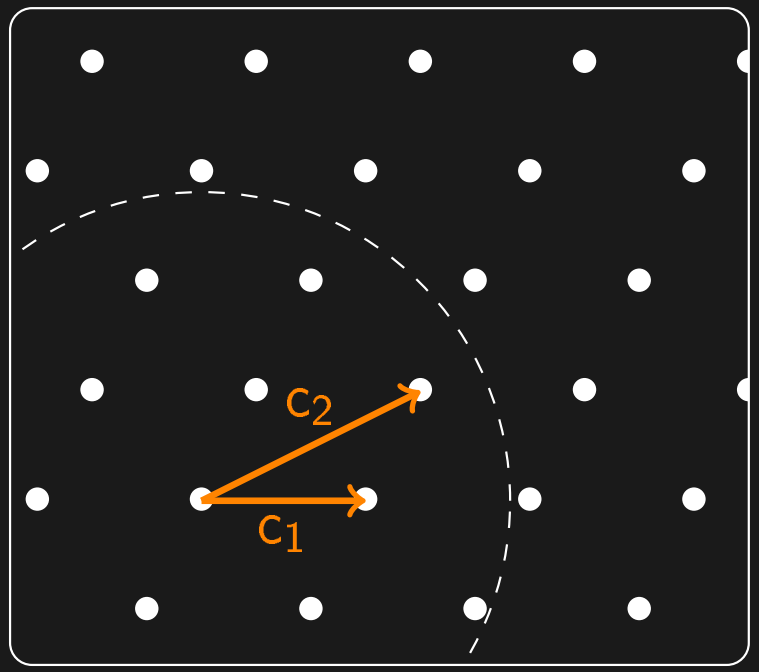


**Approximate** short basis problem

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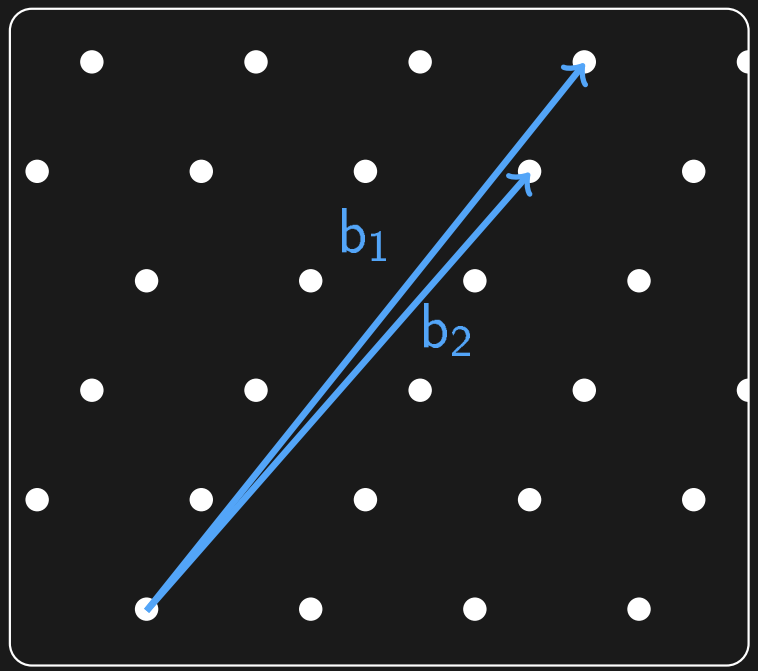


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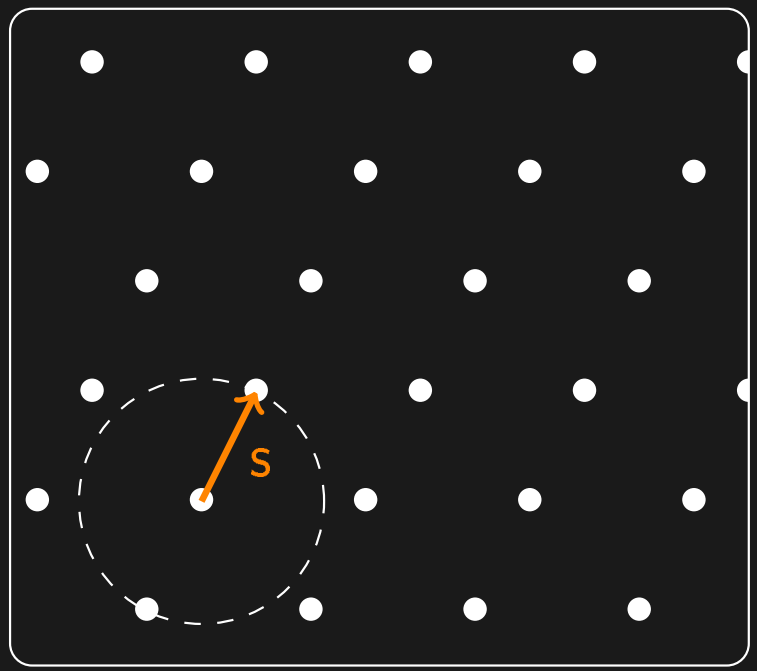


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$$\|s\| \leq \min_{v \in \mathcal{L} \setminus \{0\}} \|v\|$$

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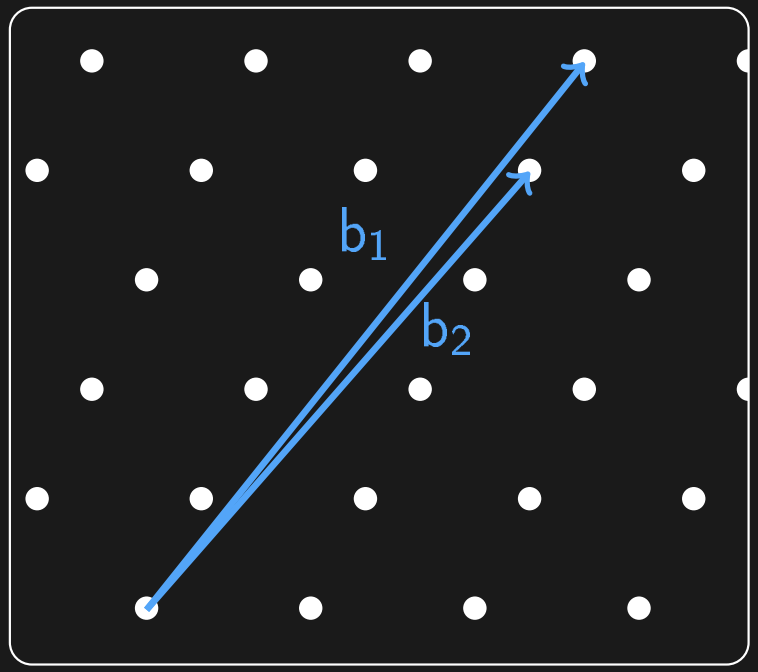


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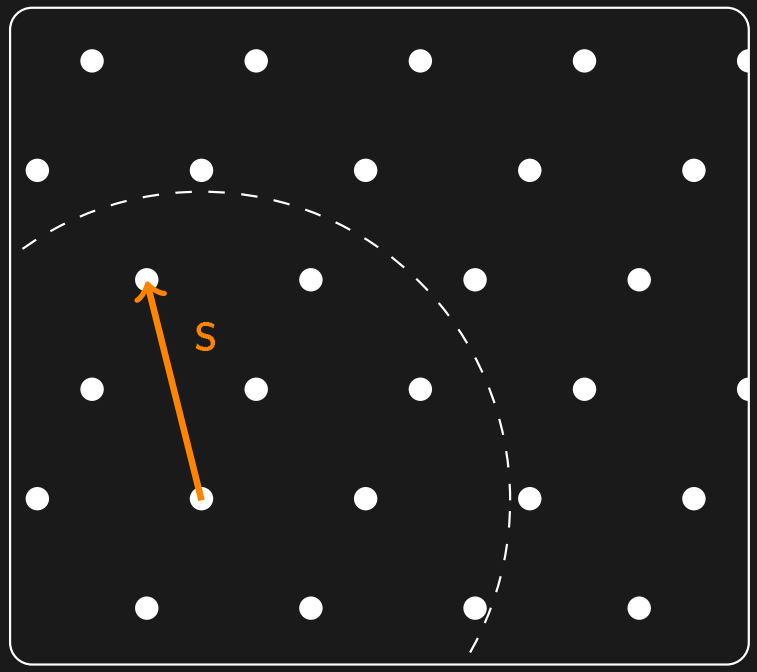


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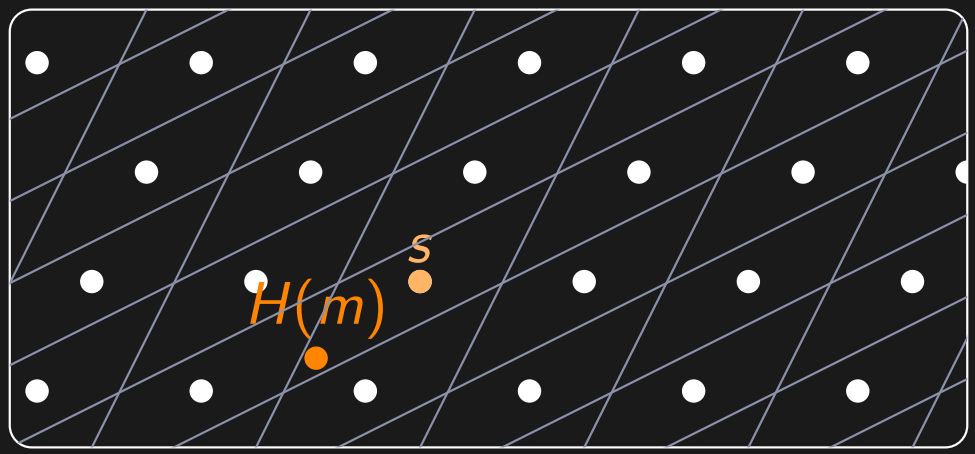
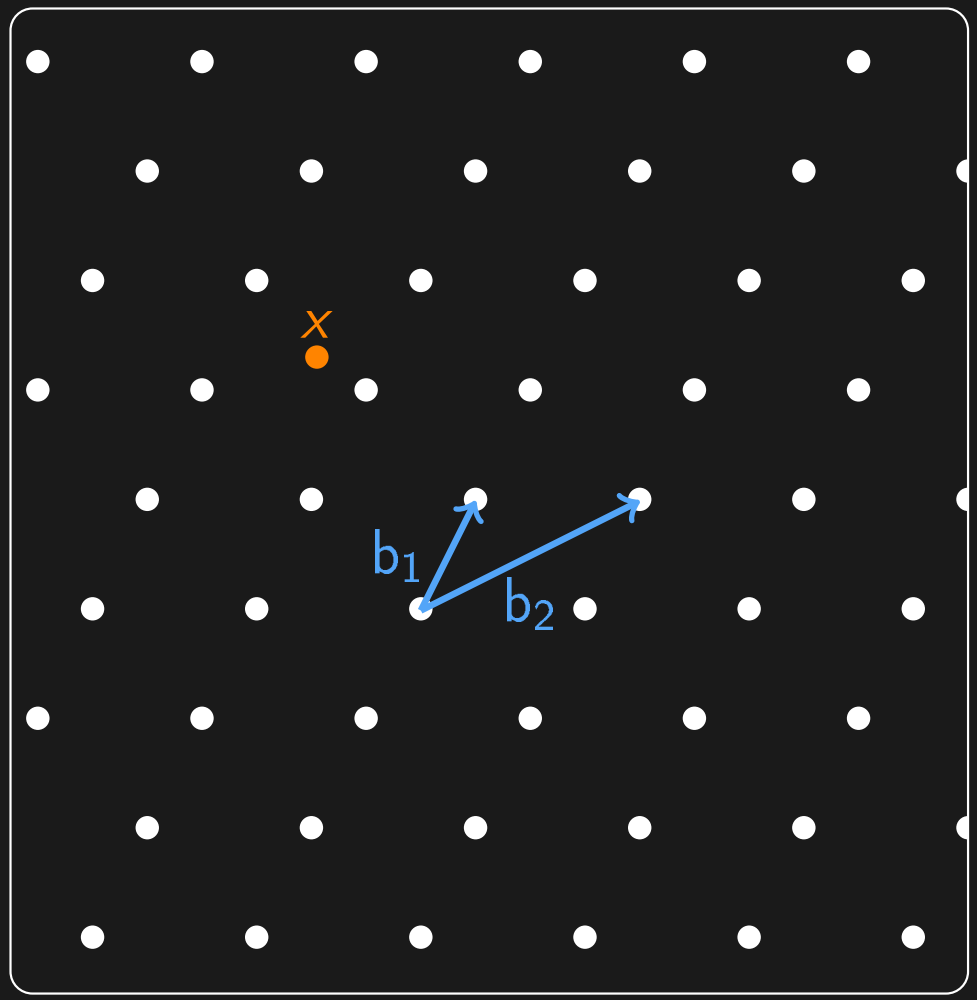
# Hash-and-sign signatures



# Finding a close vector using a short basis




Input: $x = 3.7 \cdot \mathbf{b}_1 - 1.4 \cdot \mathbf{b}_2$

Objective: find $s \in \mathcal{L}$ close to $x$

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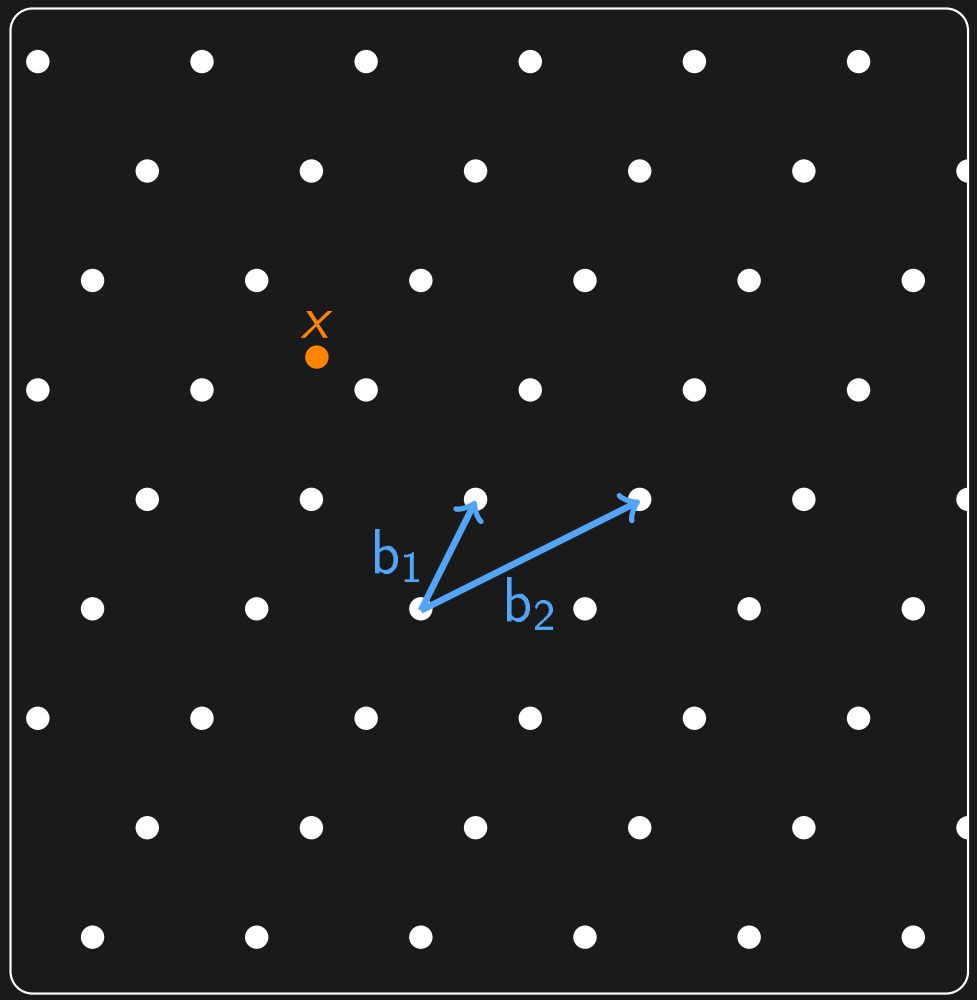


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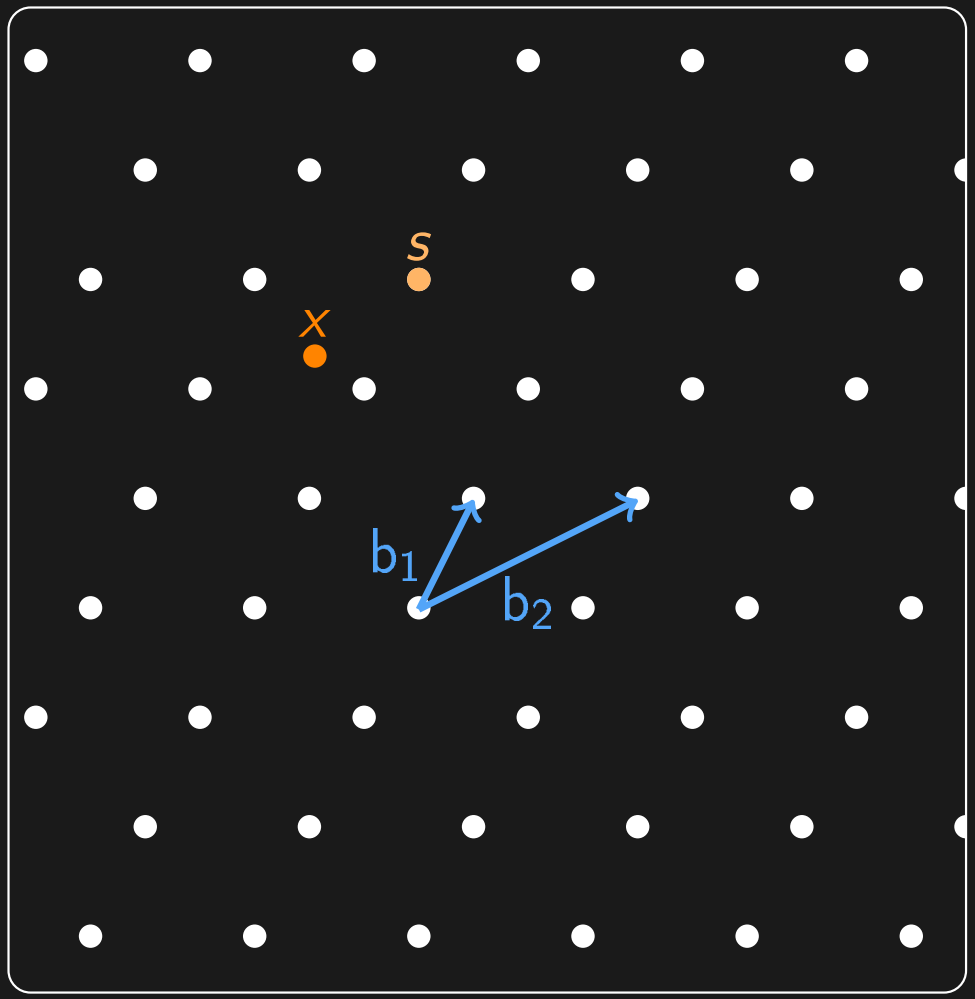


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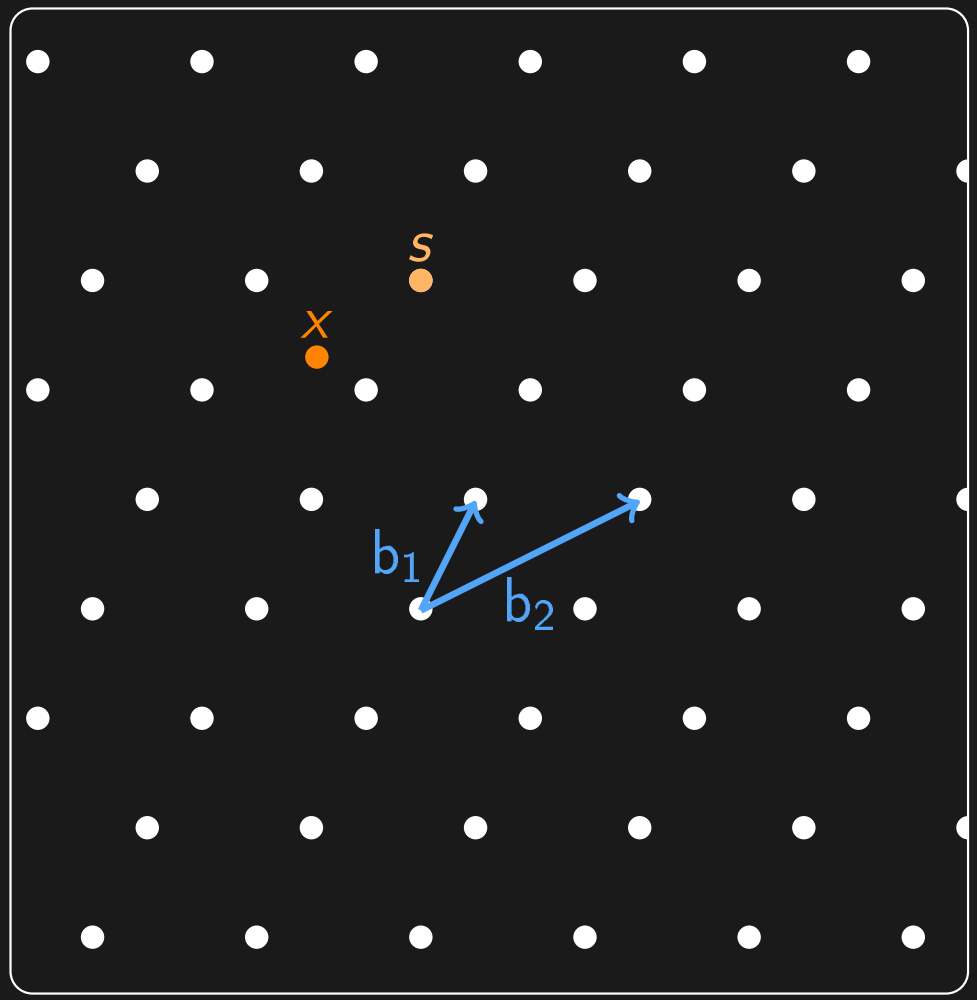


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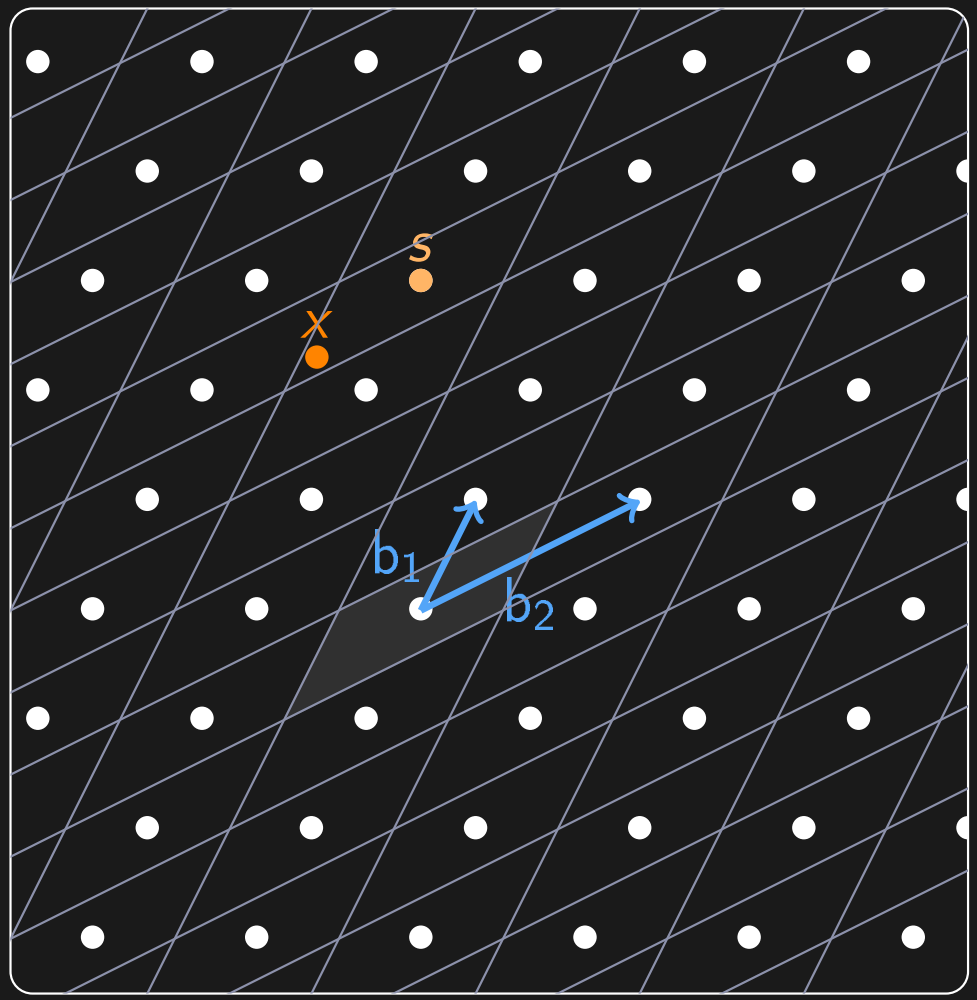


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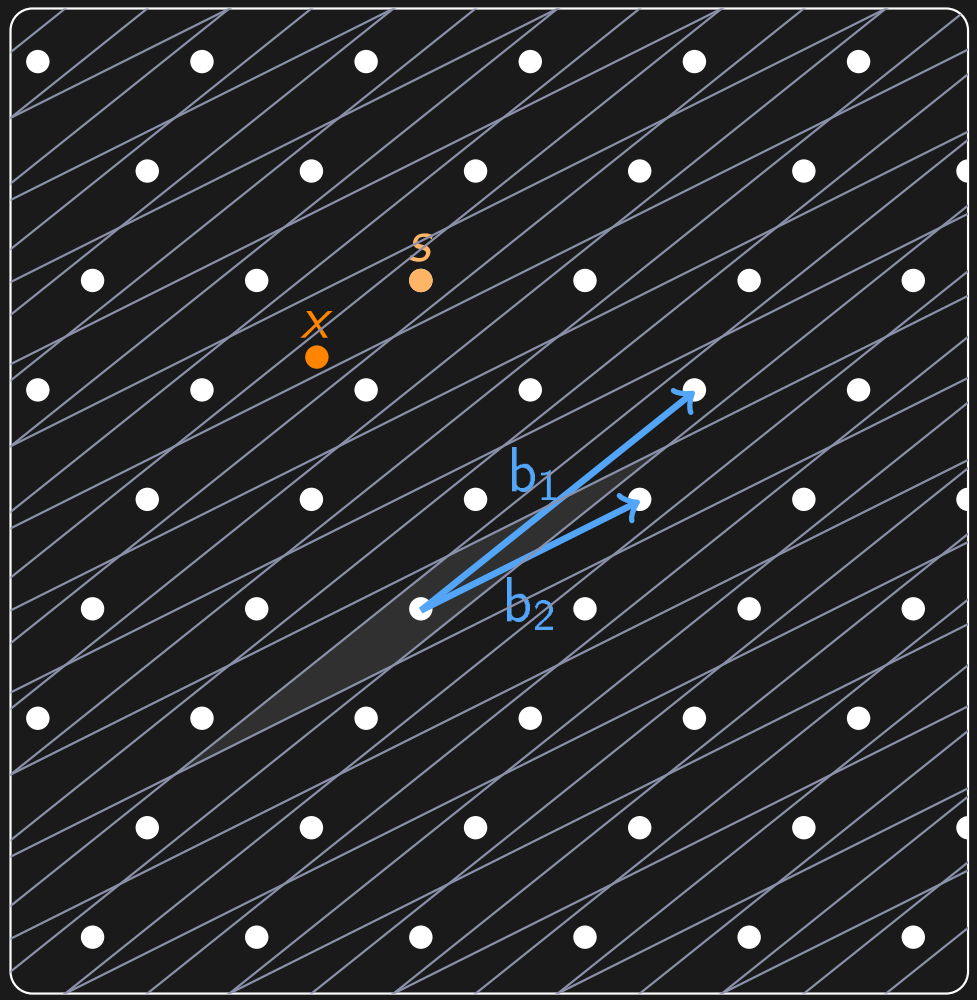


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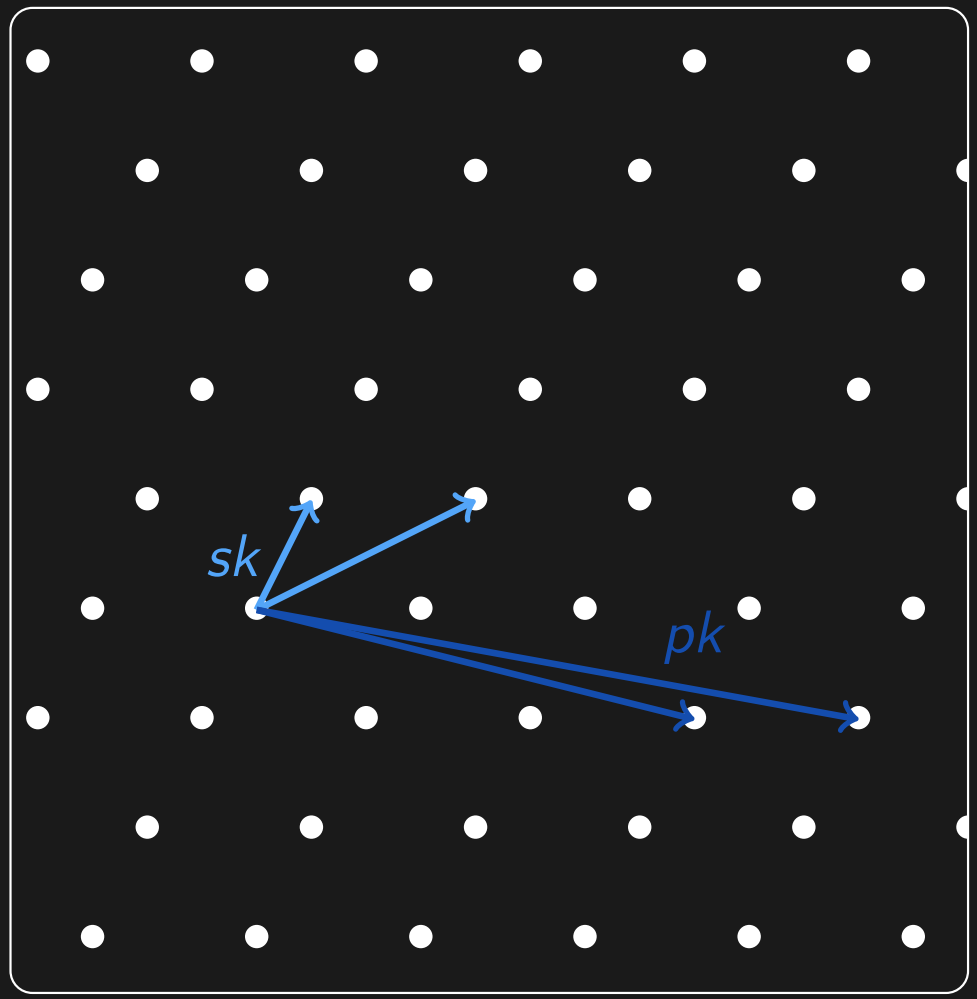


KeyGen:

- $pk$ = bad basis of $\mathcal{L}$
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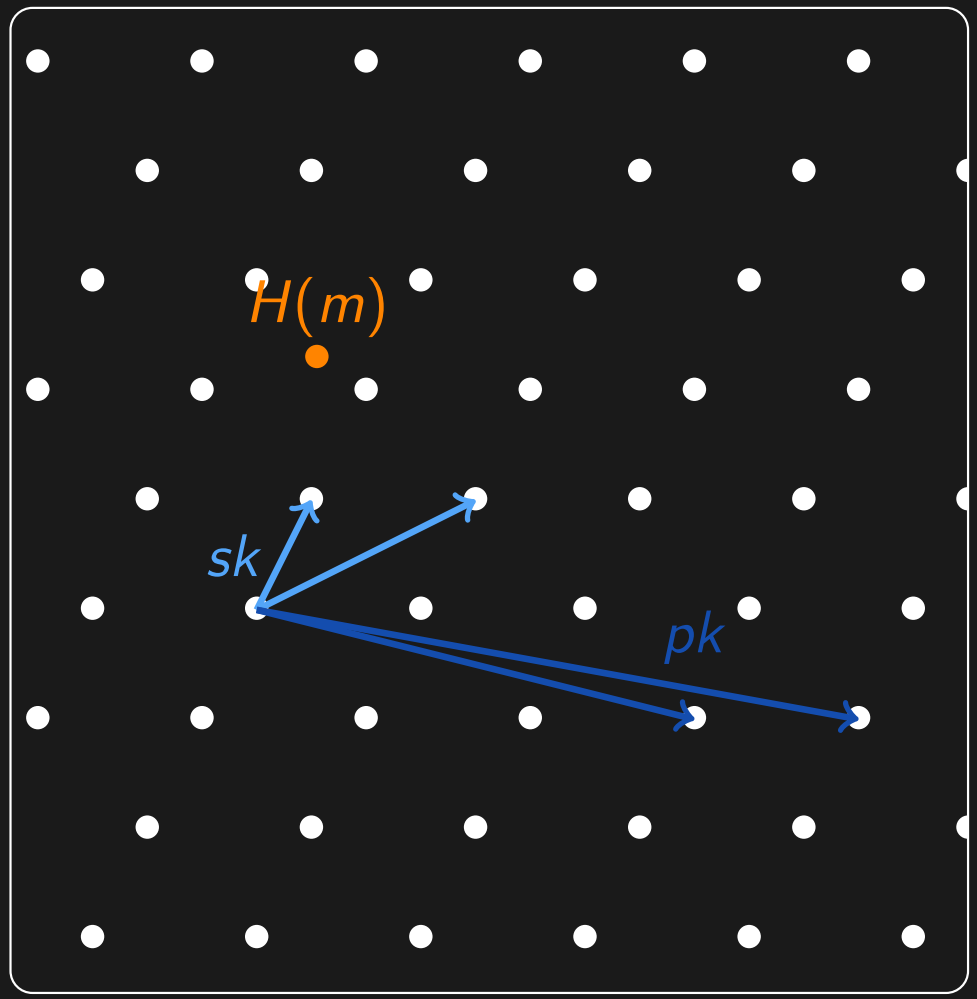


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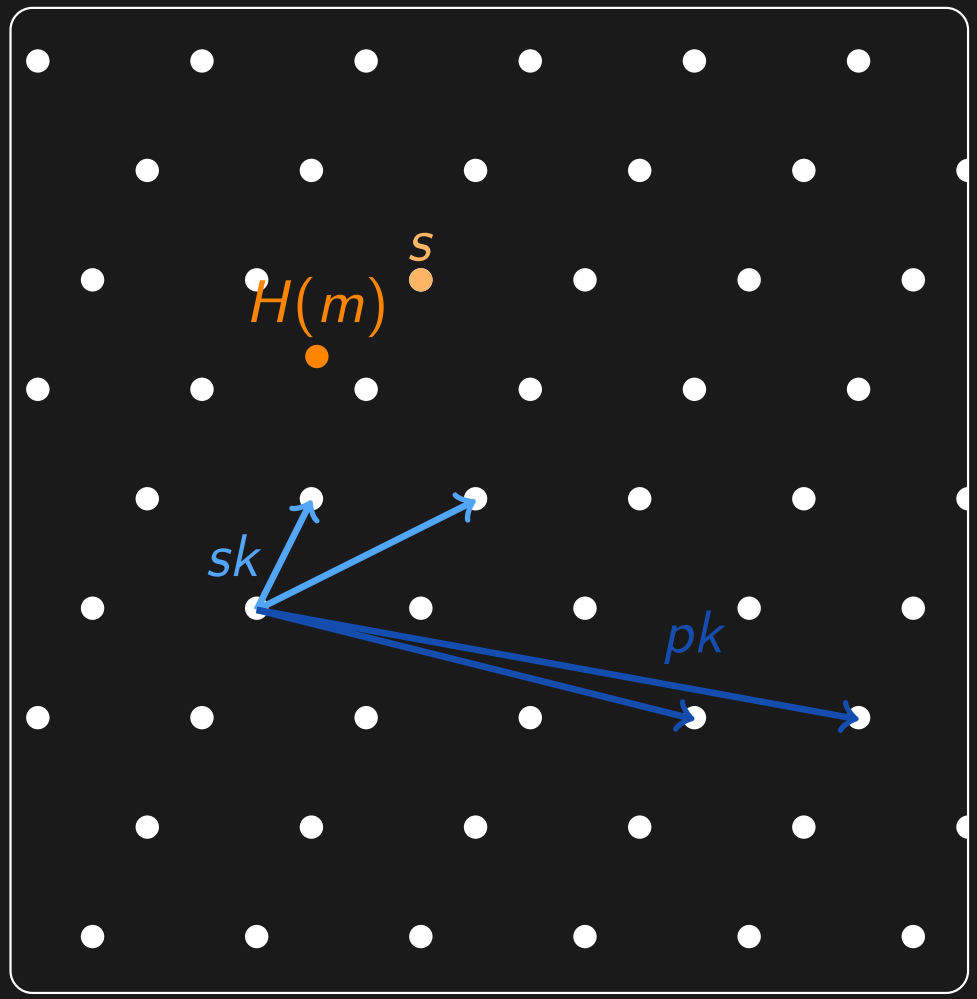


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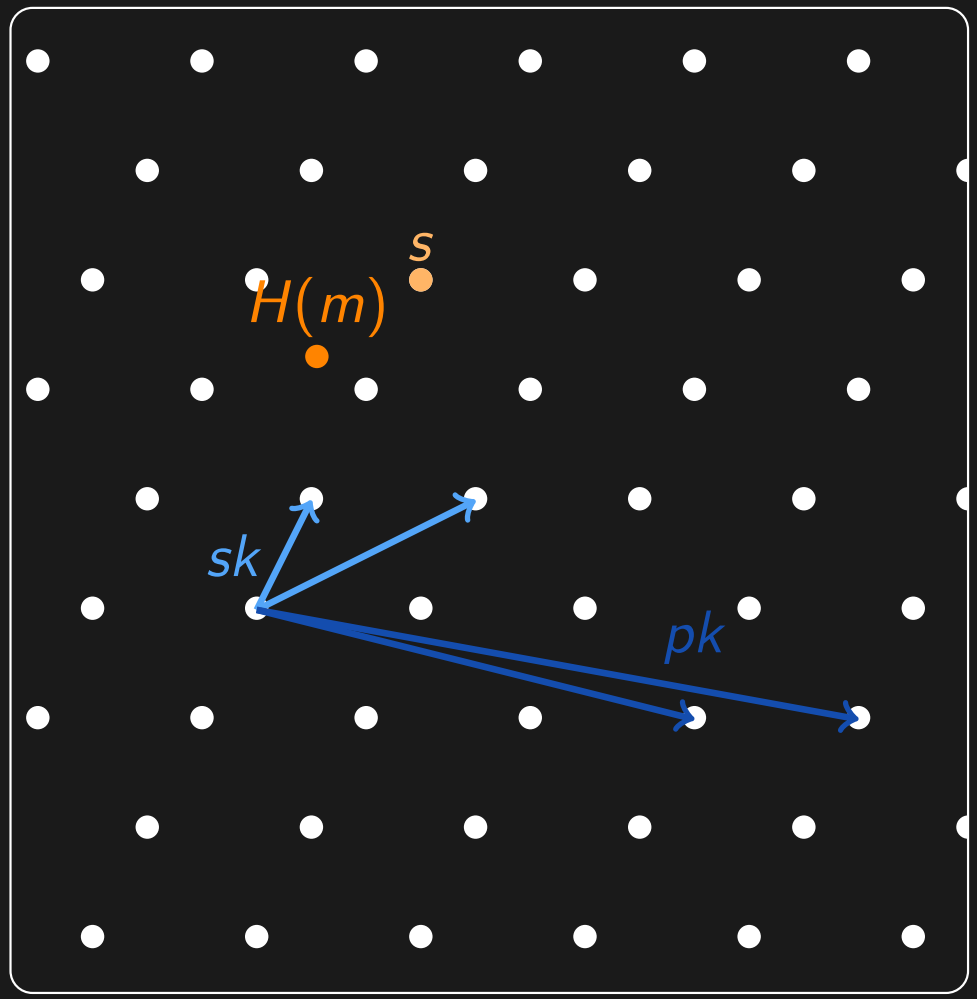


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Verify$(m, s, pk)$:

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

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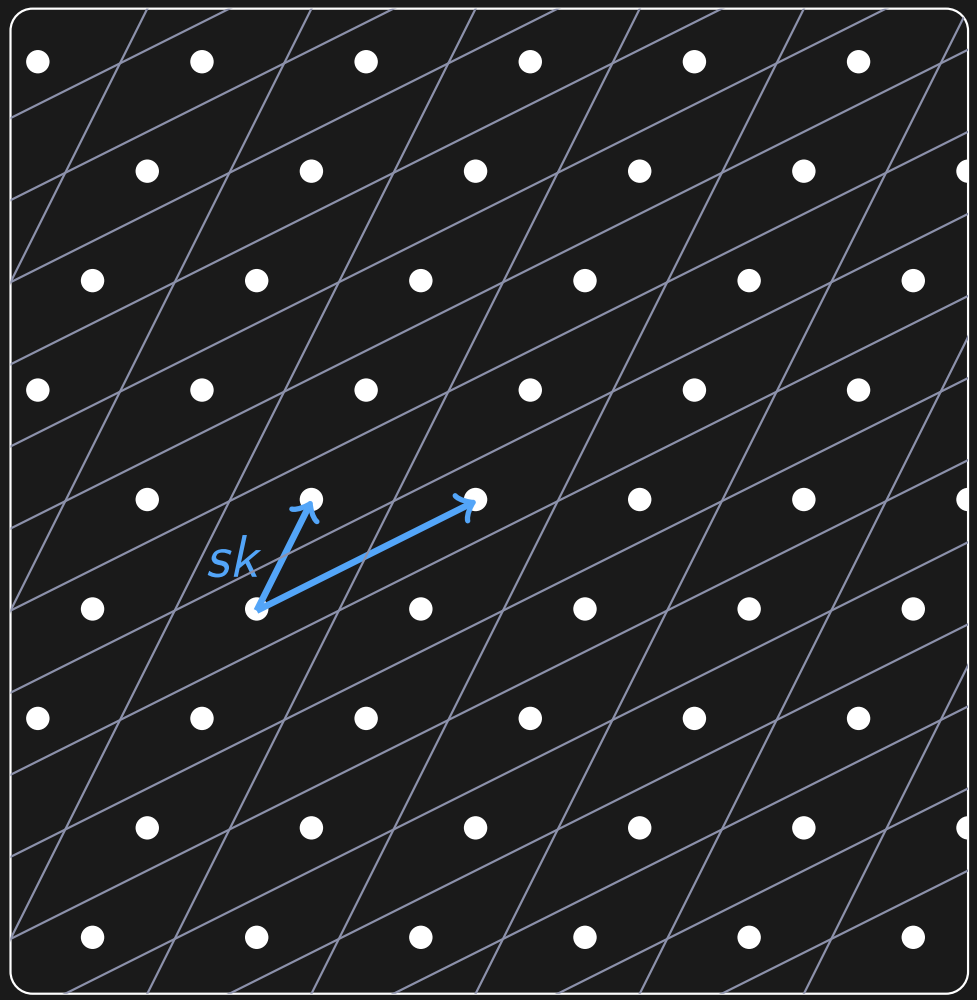
# Attack on this first idea [NR06]




Parallelepiped attack:

[NR06] Nguyen and Regev. Learning a parallelepiped: Cryptanalysis of GGH and NTRU signatures. J. Cryptology

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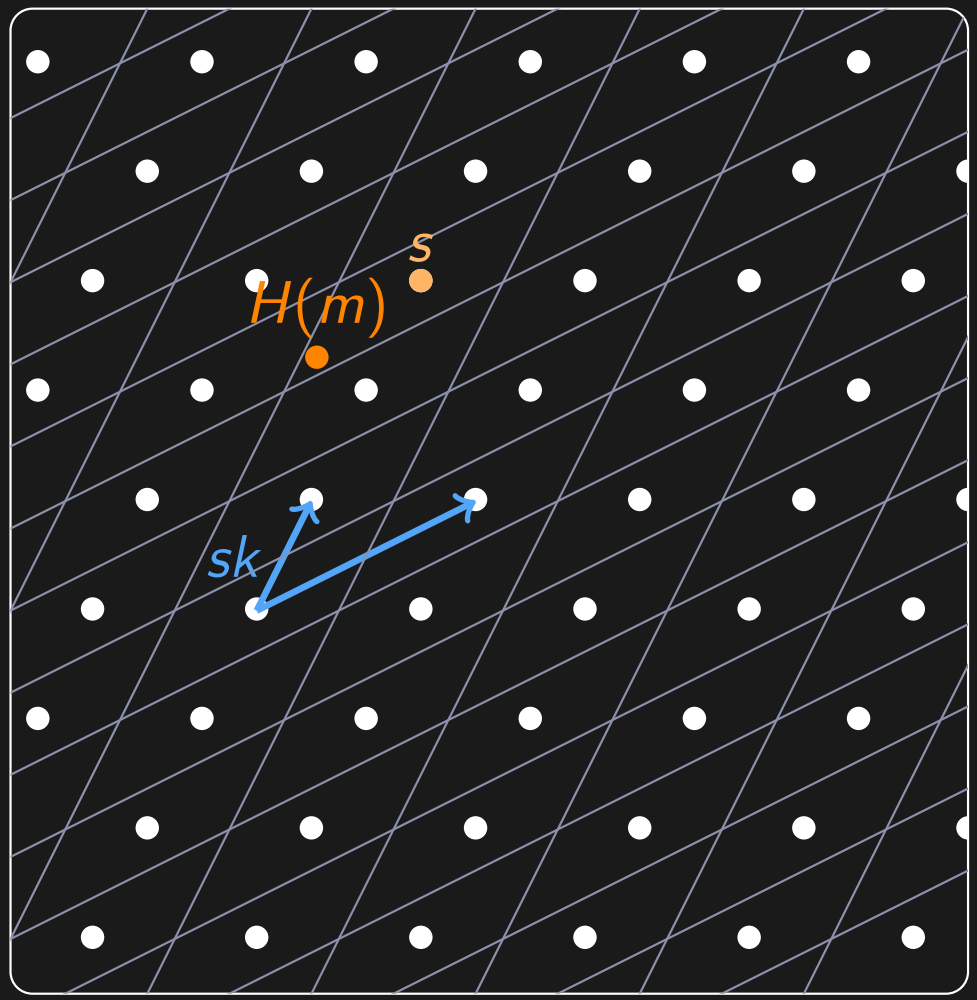


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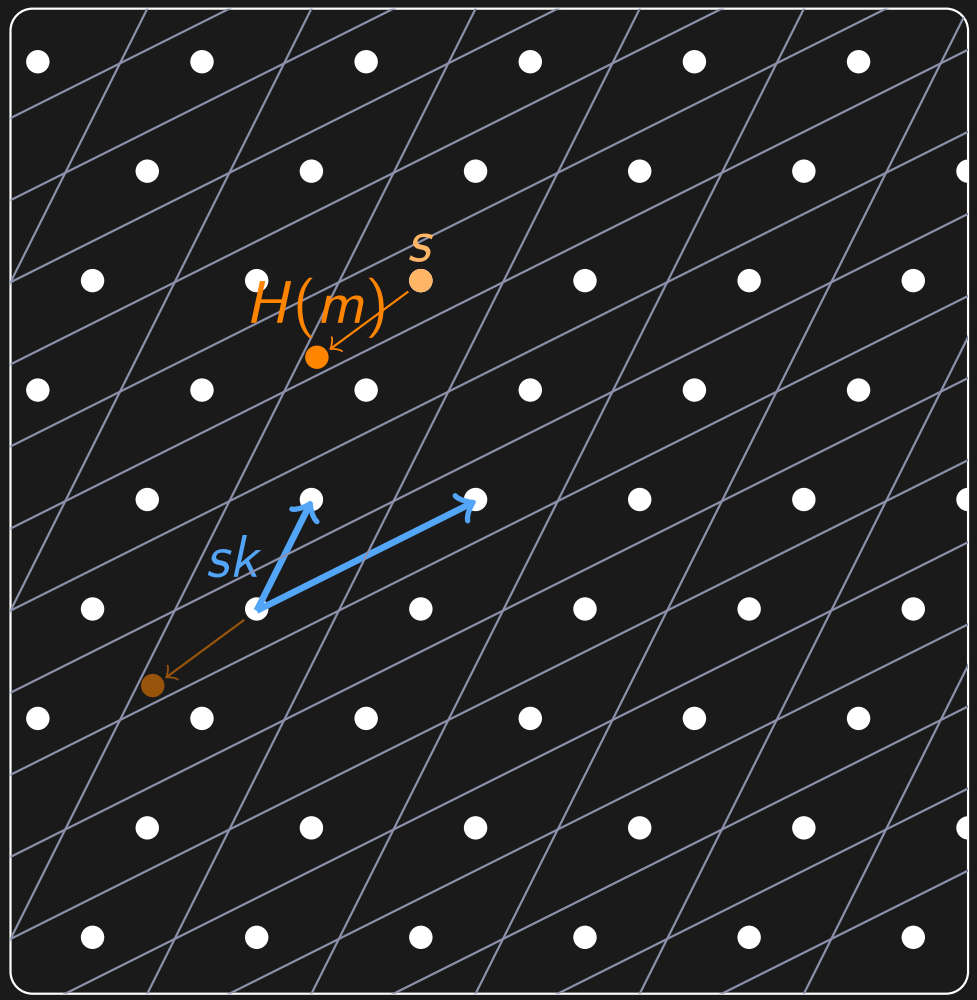


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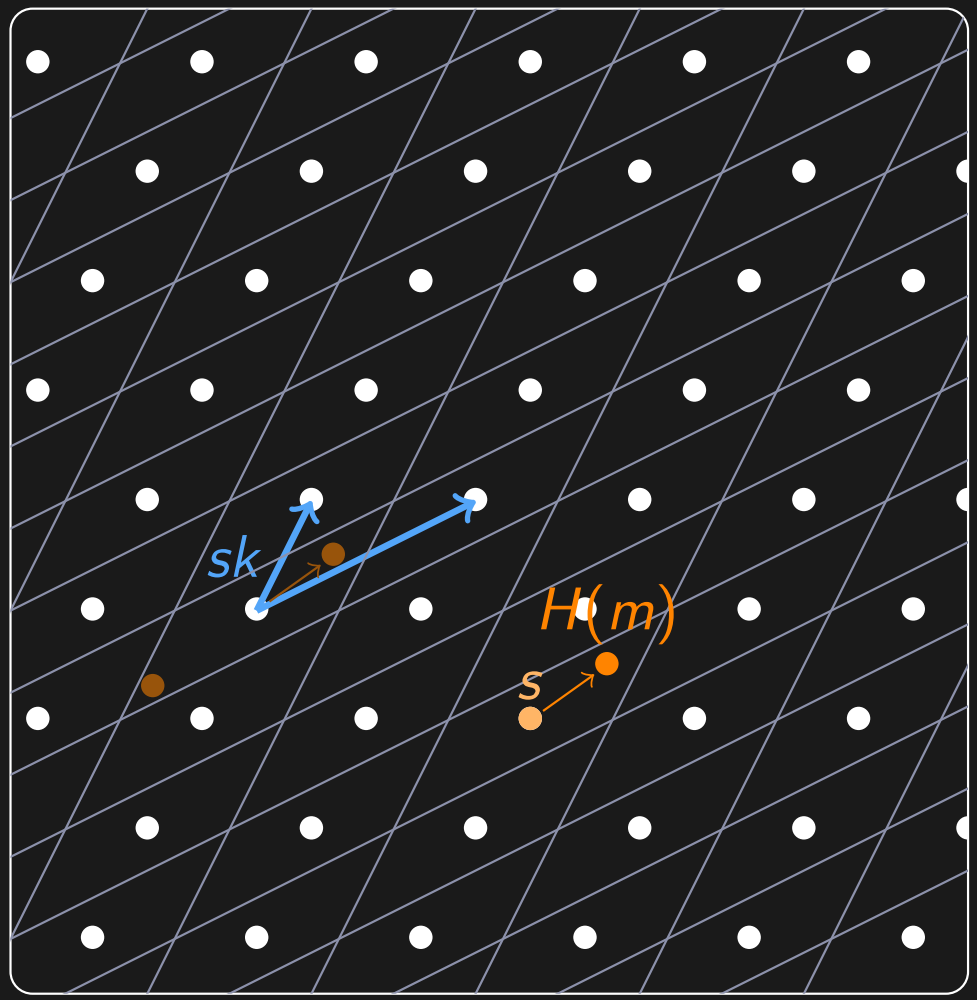


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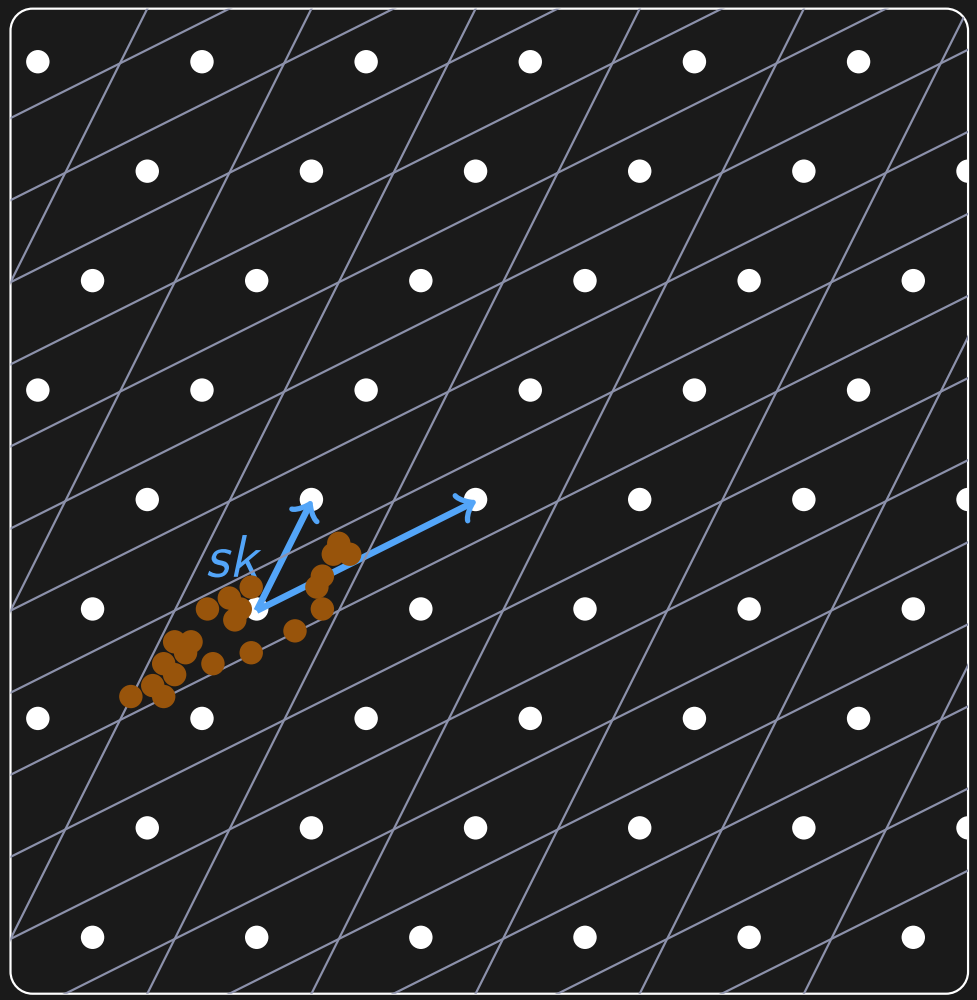


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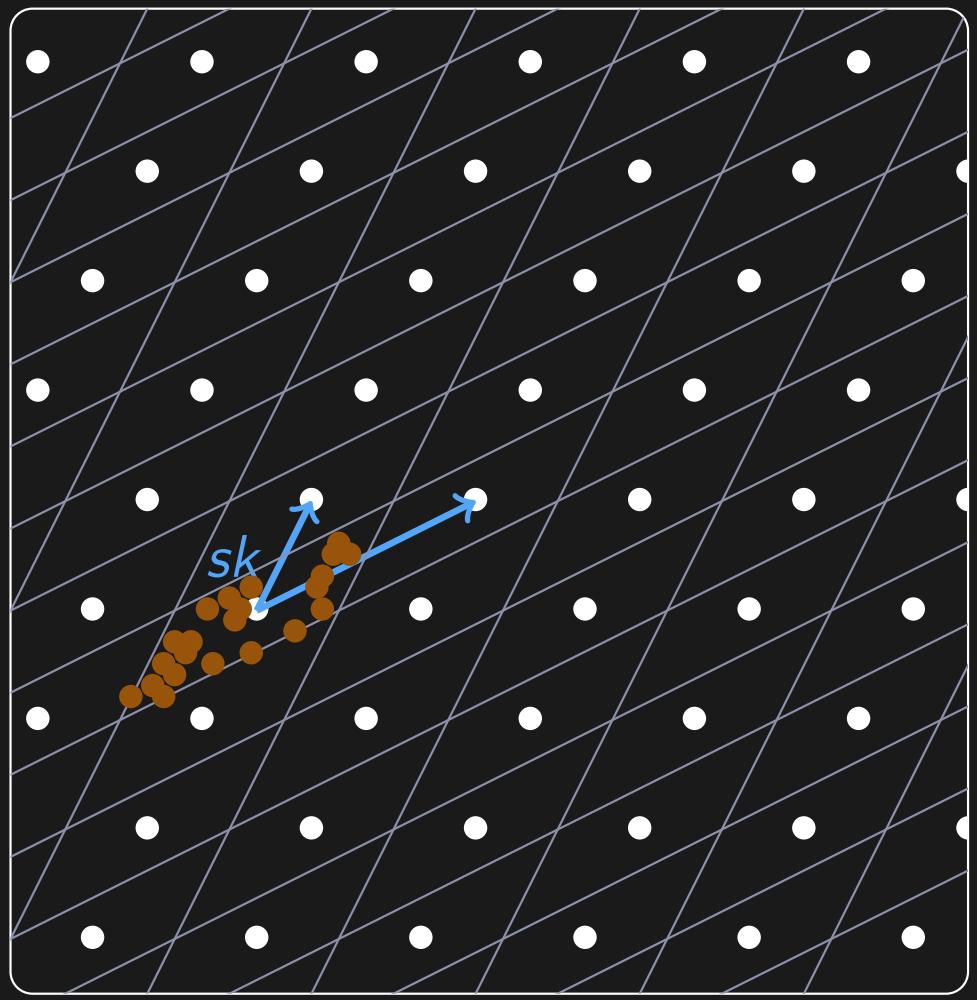


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From the shape of the parallelepiped, one can recover the short basis

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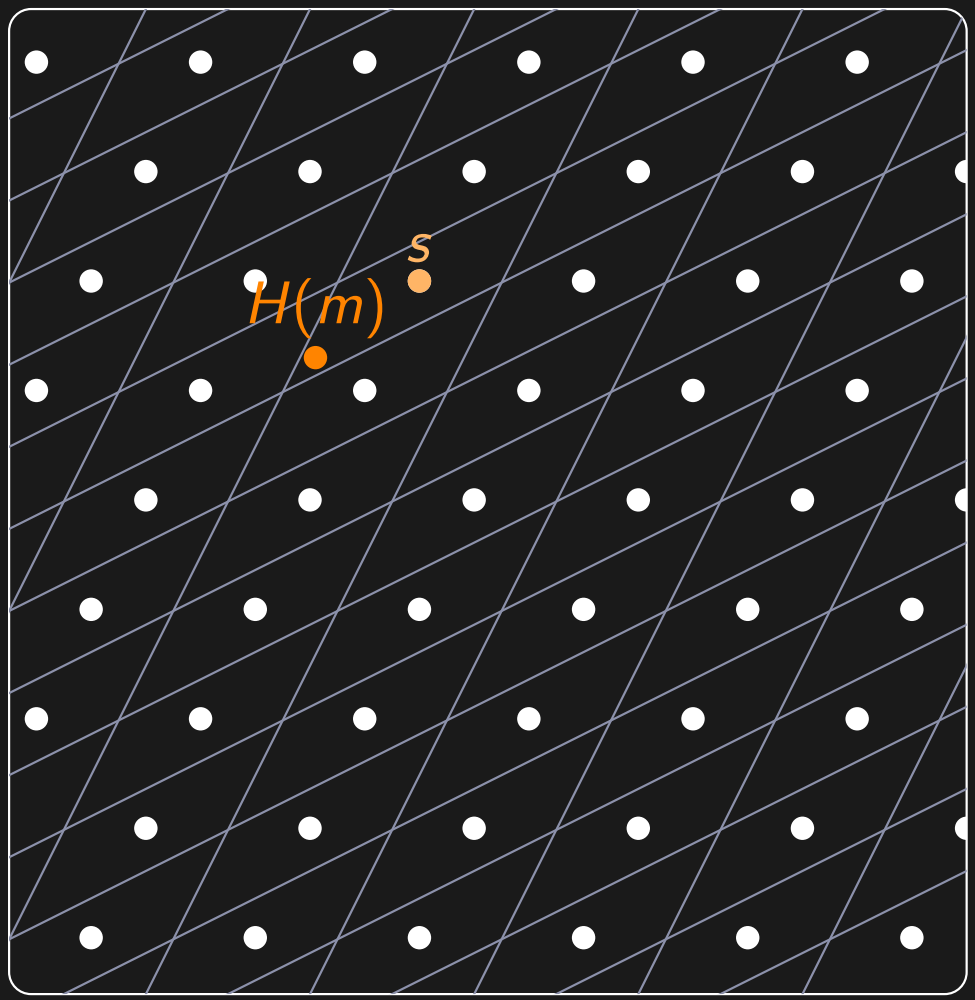
# Preventing the attack [GPV08]




Idea: do not decode deterministically but randomly

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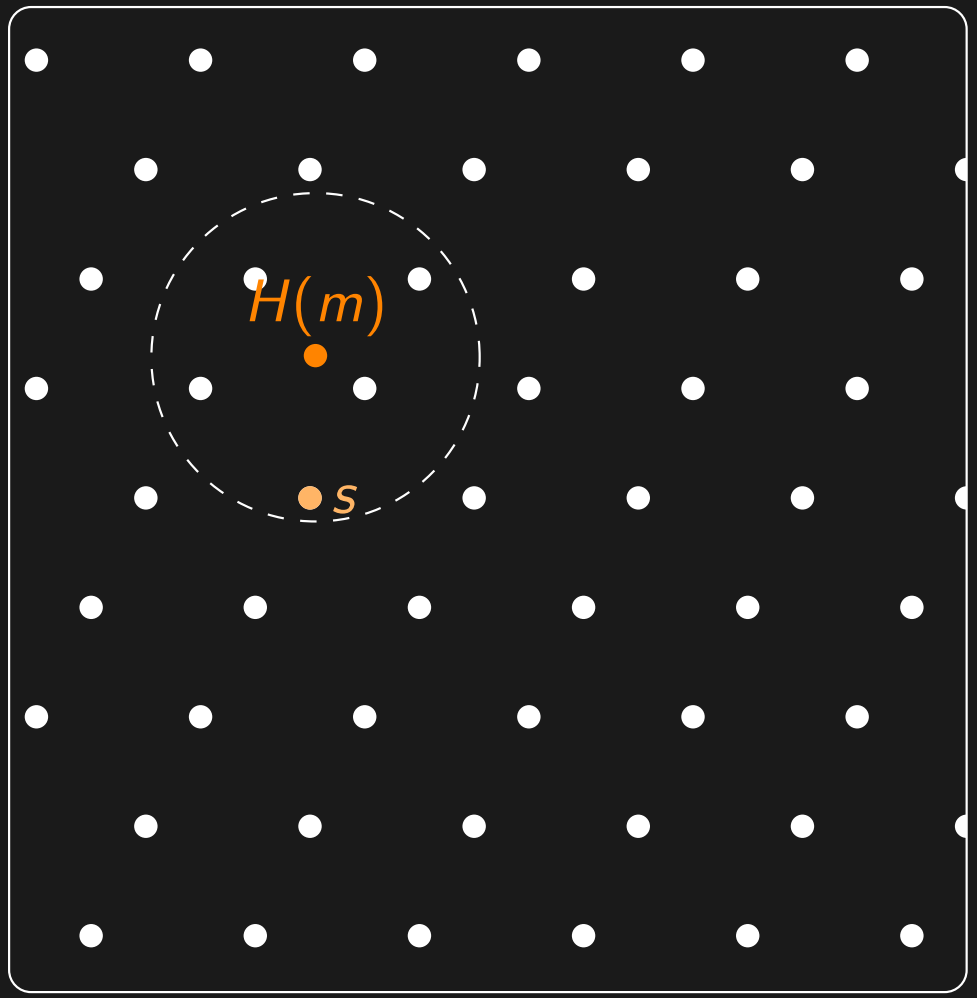


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Sign$(m, sk)$:

- $x = H(m)$ (hash the message)
- sample $s \in \mathcal{L} \cap \mathcal{B}_r(x)$
  (small radius $r \approx \max_i \|b_i\|$,
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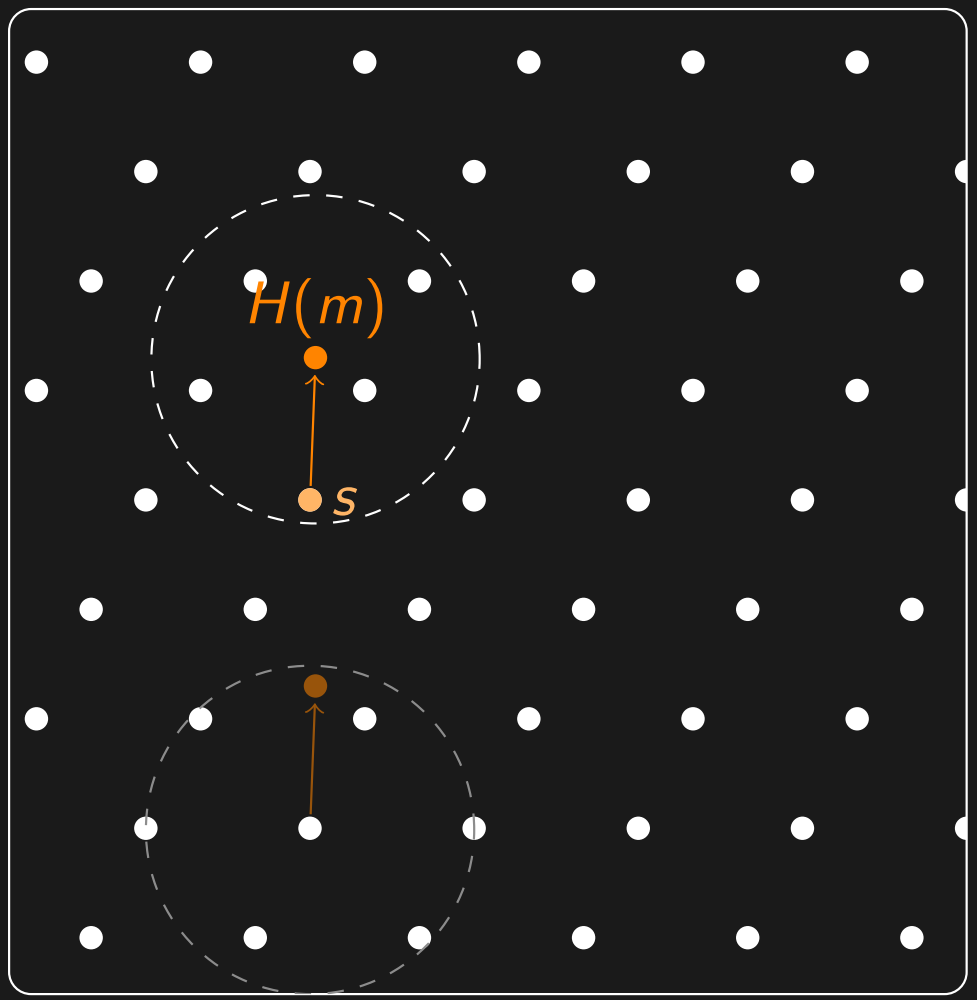


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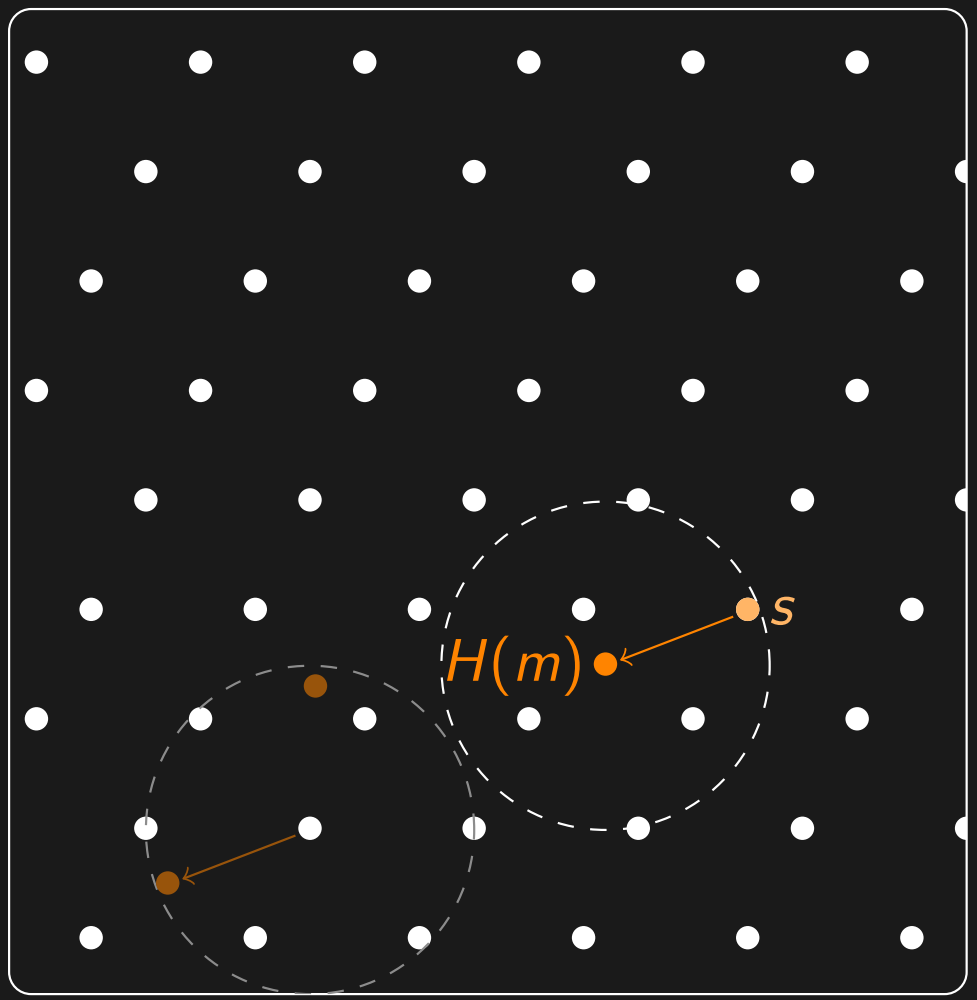


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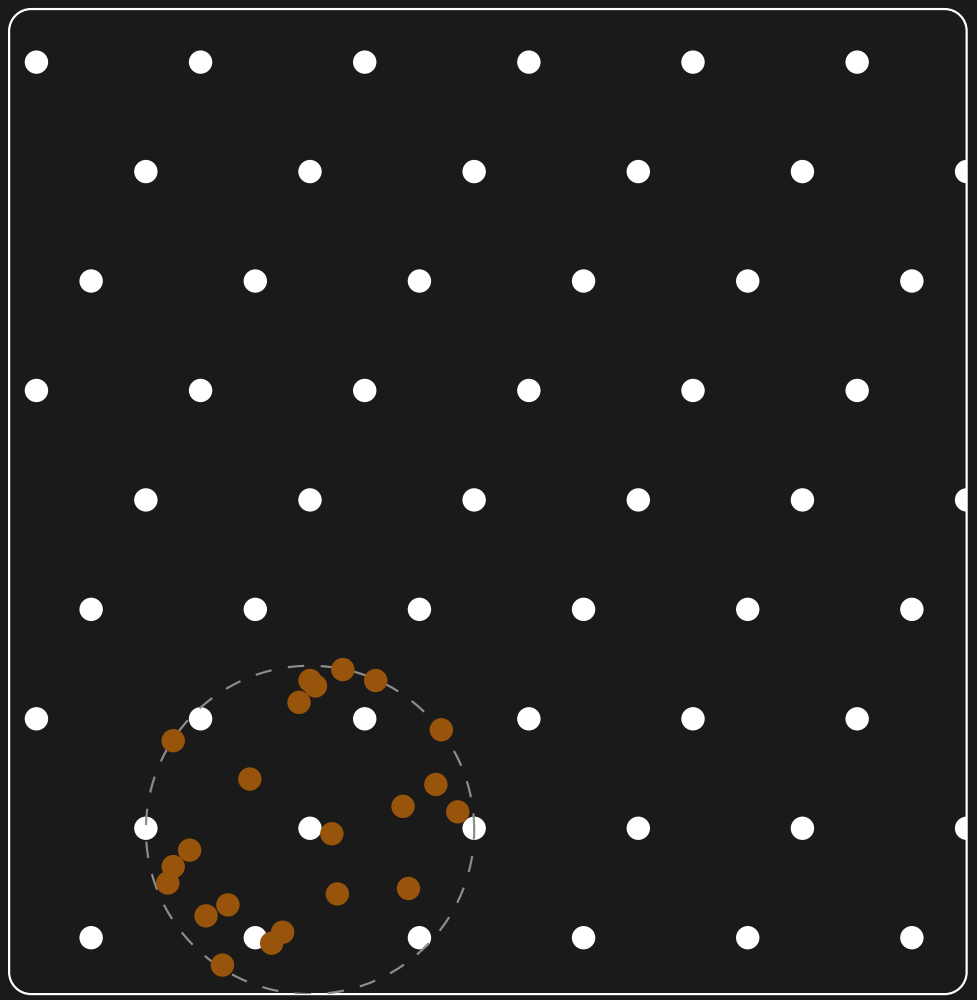

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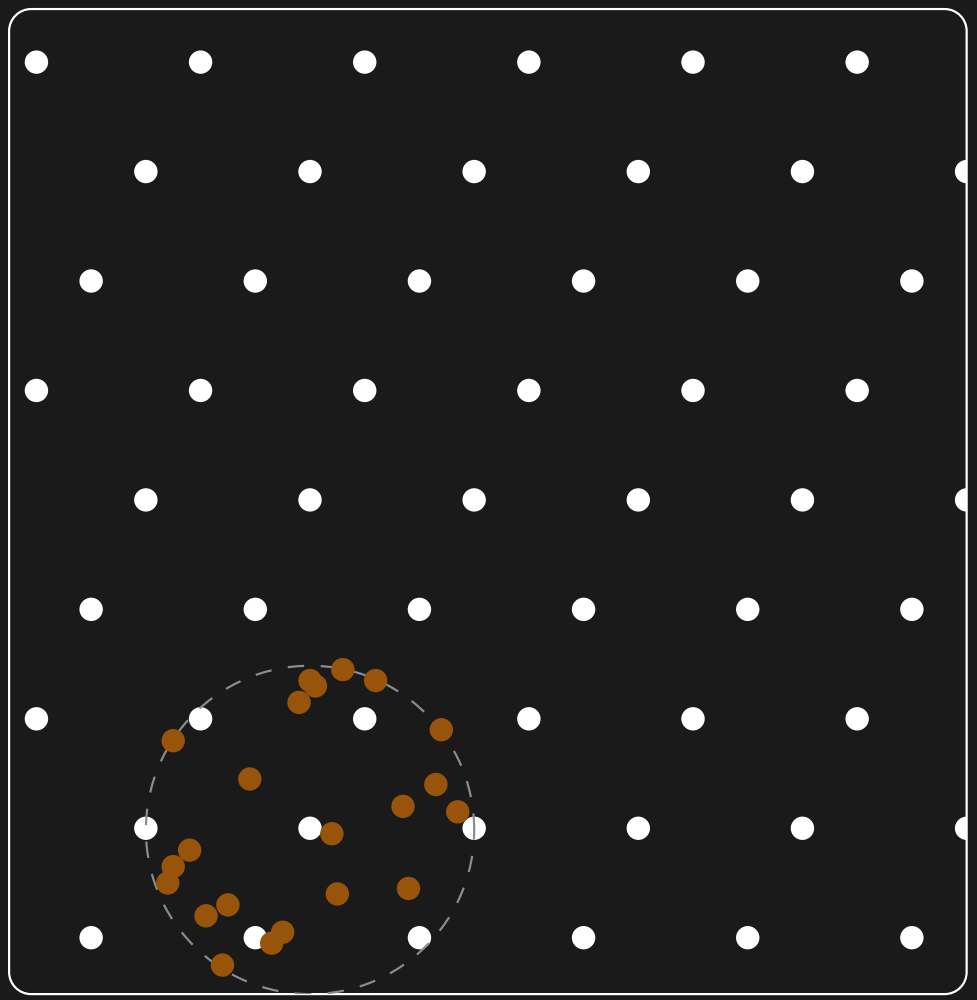

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Lemma: if an adversary can forge signatures, then she can recover a short basis of $\mathcal{L}$ using only $\mathrm{pk}$
(in the ROM)

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# Instantiating the framework

Hash-and-sign framework: provably secure if we have a hard lattice $\mathcal{L}$

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We need: a way to generate random hard lattices

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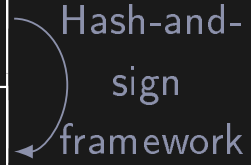
| Assumption | SIS (short integer solution) | | |
|---|---|---|---|
| Construction's name | GPV | | |




[Ajt96] Ajtai. Generating hard instances of lattice problems. STOC.

[GPV08] Gentry, Peikert, Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC

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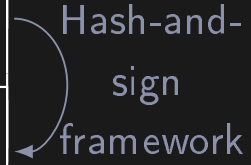


[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

[Falcon] Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Prest, Ricosset, Seiler, Whyte, Zhang. NIST standard

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|---|---|---|---|
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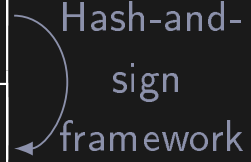
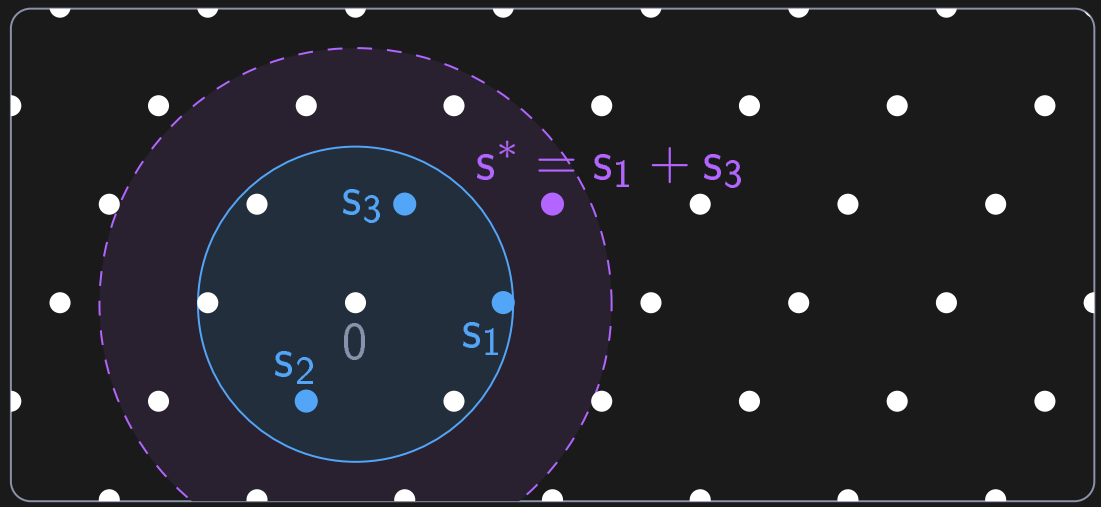



[DW22] Ducas and van Woerden. On the lattice isomorphism problem, quadratic forms [...]. Eurocrypt

[DPPW23] Ducas, Postlethwaite, Pulles, van Woerden. Hawk: Module LIP makes lattice signatures [...]. Asiacrypt

# Attack on a knowledge variant of SIS



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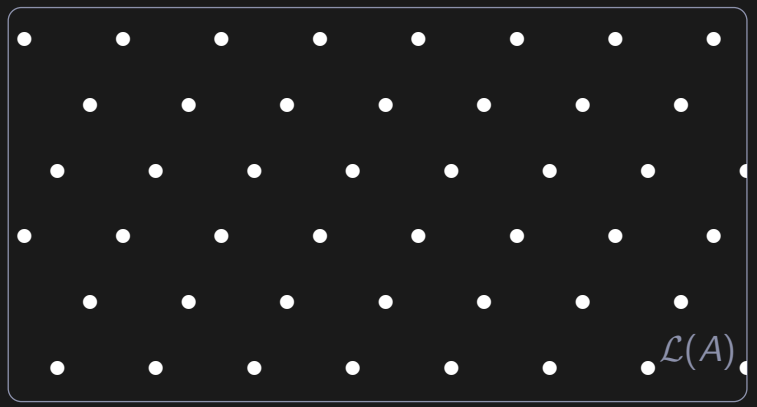


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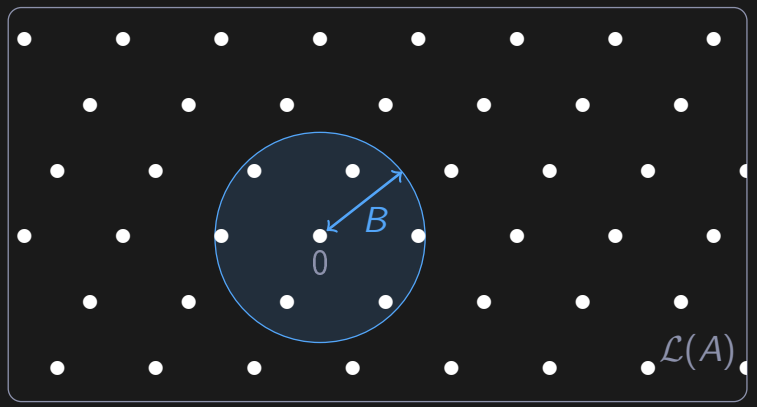


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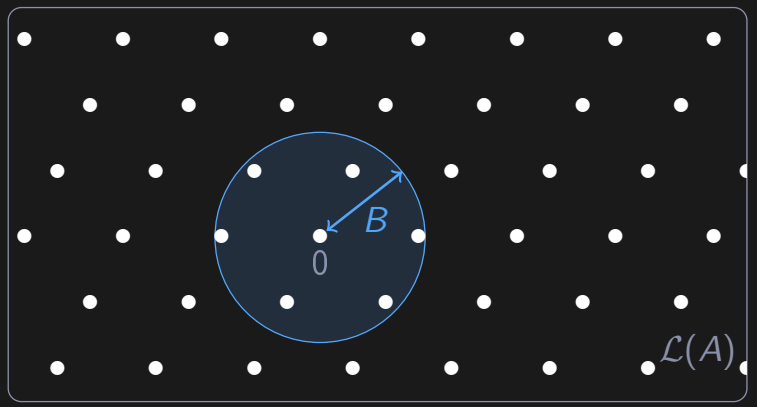


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## SIS assumption

Solving the SIS problem is hard

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# Knowledge variant of SIS (adapted from [ACLMT22, Alb22])

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**Parameters:** $k \geq m \geq n$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

[ACLMT22] Albrecht, Cini, Lai, Malavolta,Thyagarajan. Lattice-based SNARKs [...]. Crypto

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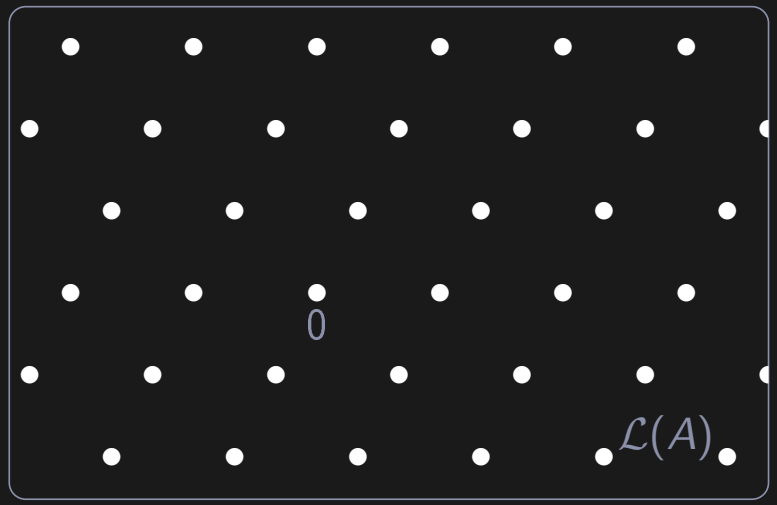



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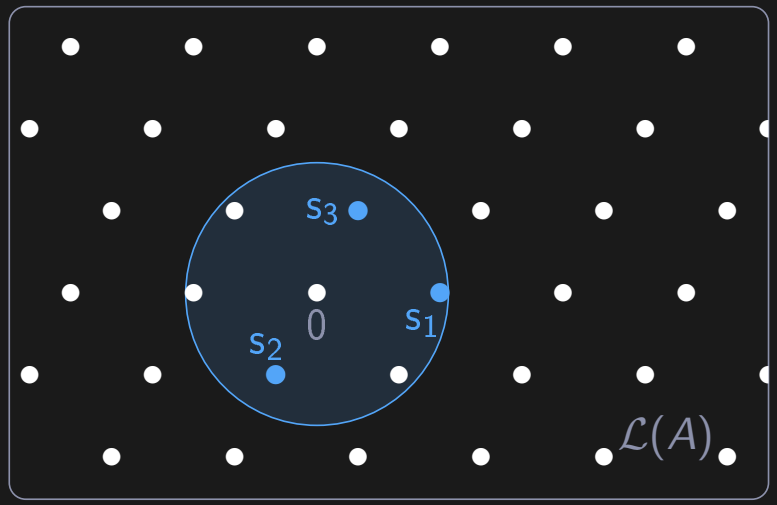



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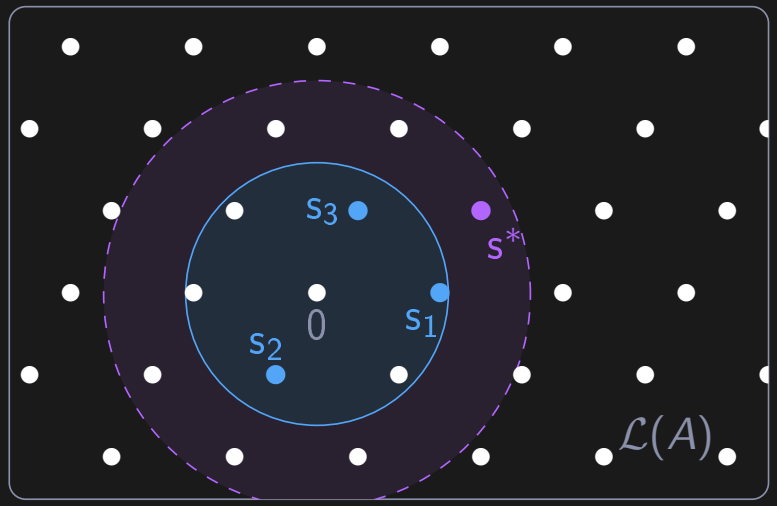



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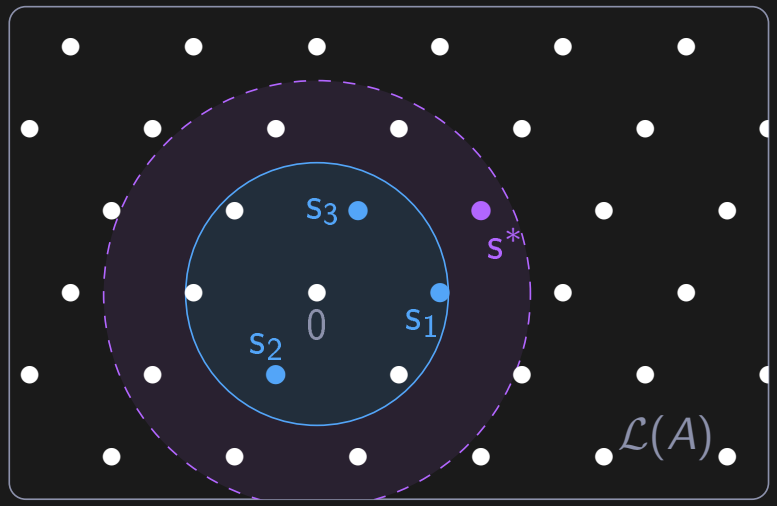


## knowledge SIS (geometric)

Given $k$ short vectors of $\mathcal{L}(A)$, the only way to produce a new short vector is by taking a small combination of these $k$ vectors.


[ACLMT22] Albrecht, Cini, Lai, Malavolta, Thyagarajan. Lattice-based SNARKs [...]. Crypto

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# Summary

## knowledge SIS assumption

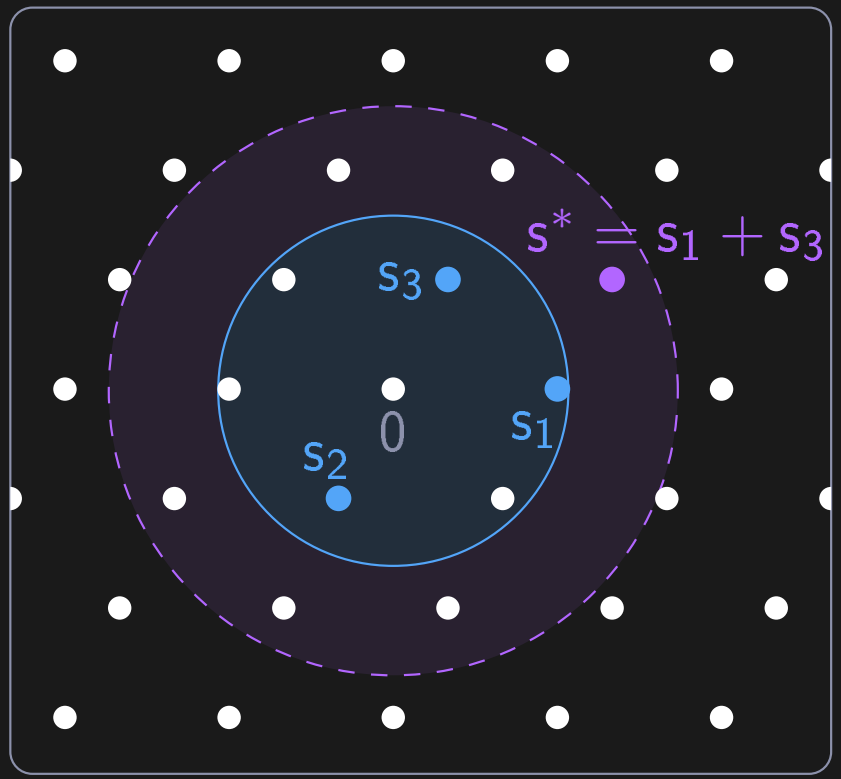
(geometric formalism)

**Parameters:** $k \geq n \geq 0$, $B > 0$

Input:

- $\mathcal{L}$ lattice of dim $n$
- $s_1, \ldots, s_k \in \mathcal{L}$ short ($\|s_i\| \leq B$)

Assumption: the only way to compute new somewhat short ($\leq \mathrm{poly}(n) \cdot B$) vectors in $\mathcal{L}$ is by taking small combinations of the $s_i$'s.



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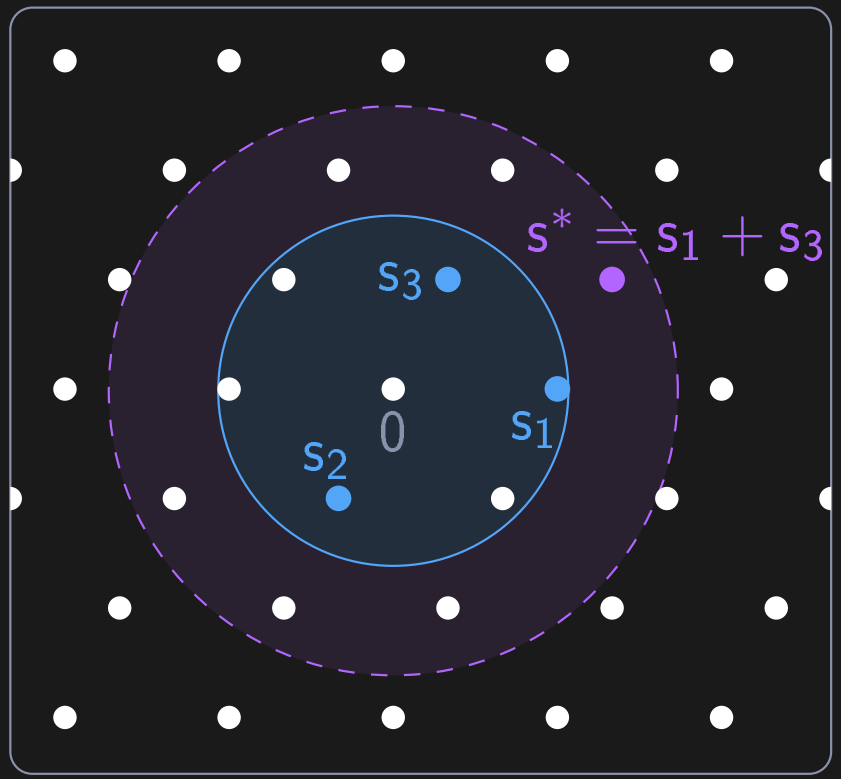


Really?

# Tool: Sampling in a lattice [PP21]

Input: center x, radius $r$

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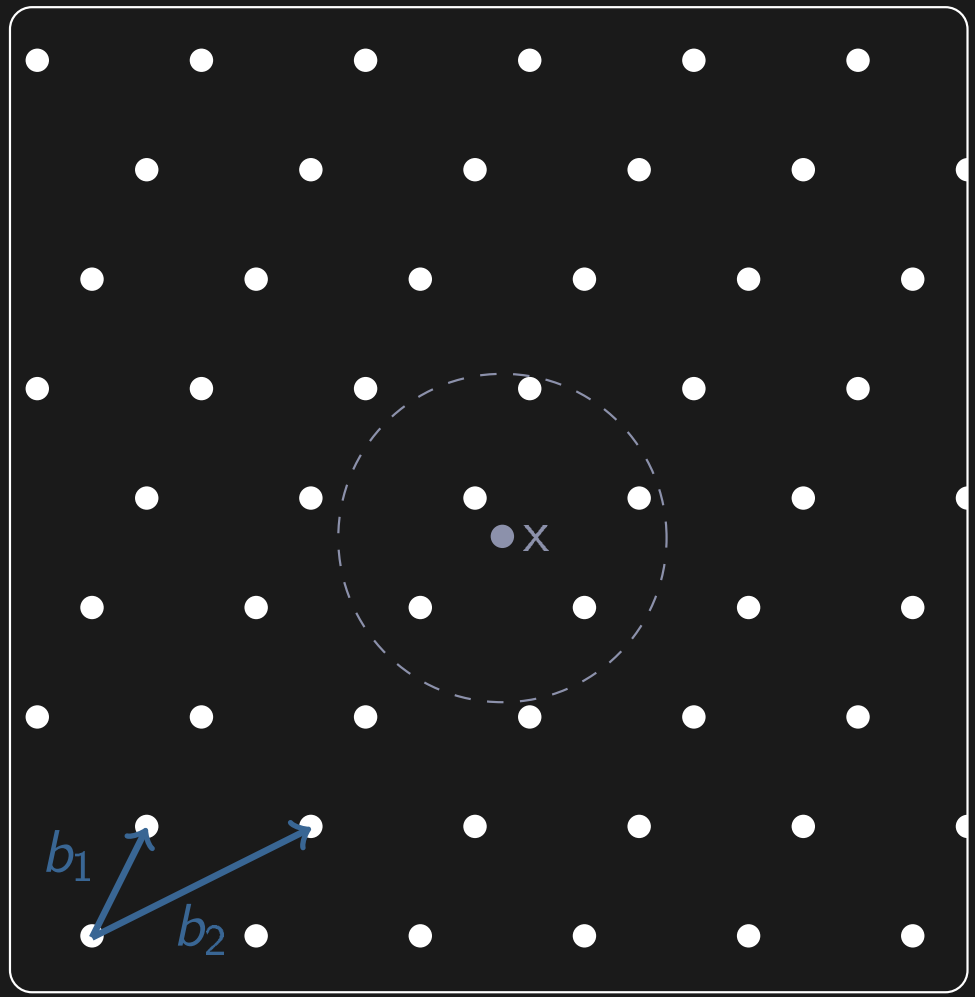
Output: $s \leftarrow \mathcal{U}(\mathcal{L} \cap \mathcal{B}_r(x))$

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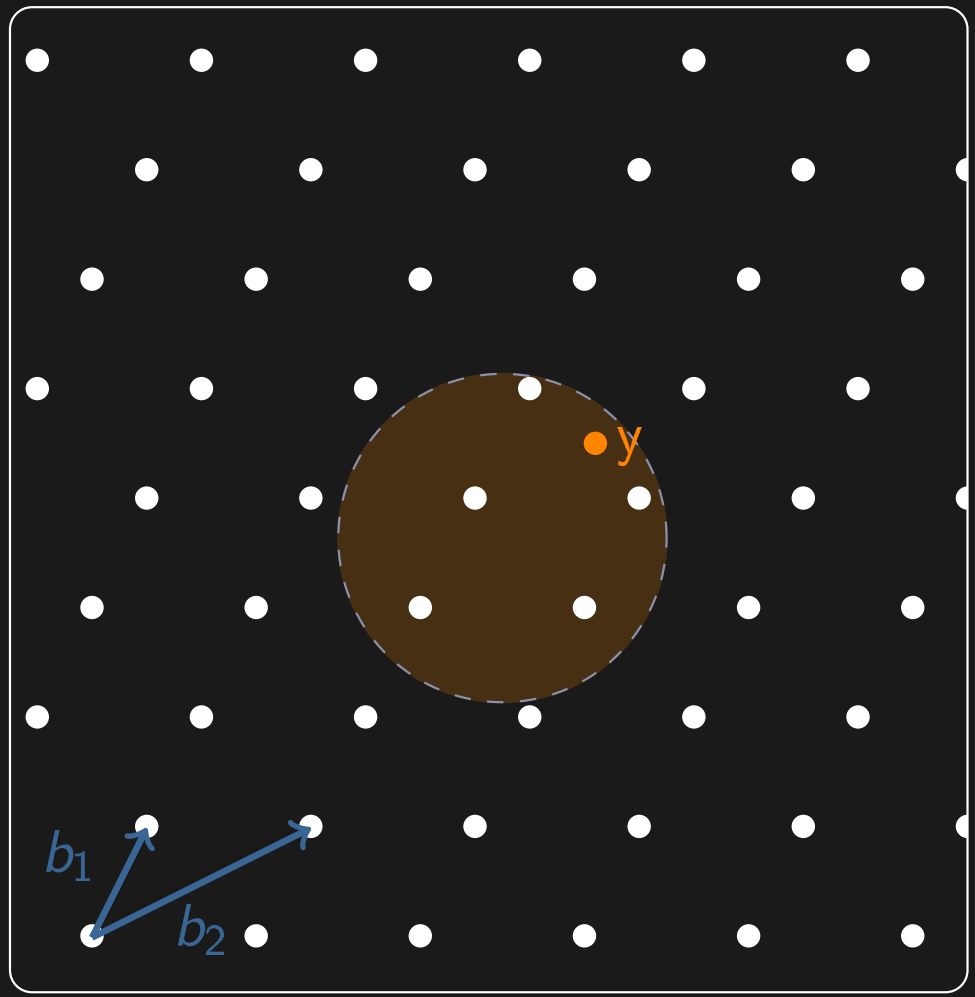
  (continuous distribution)

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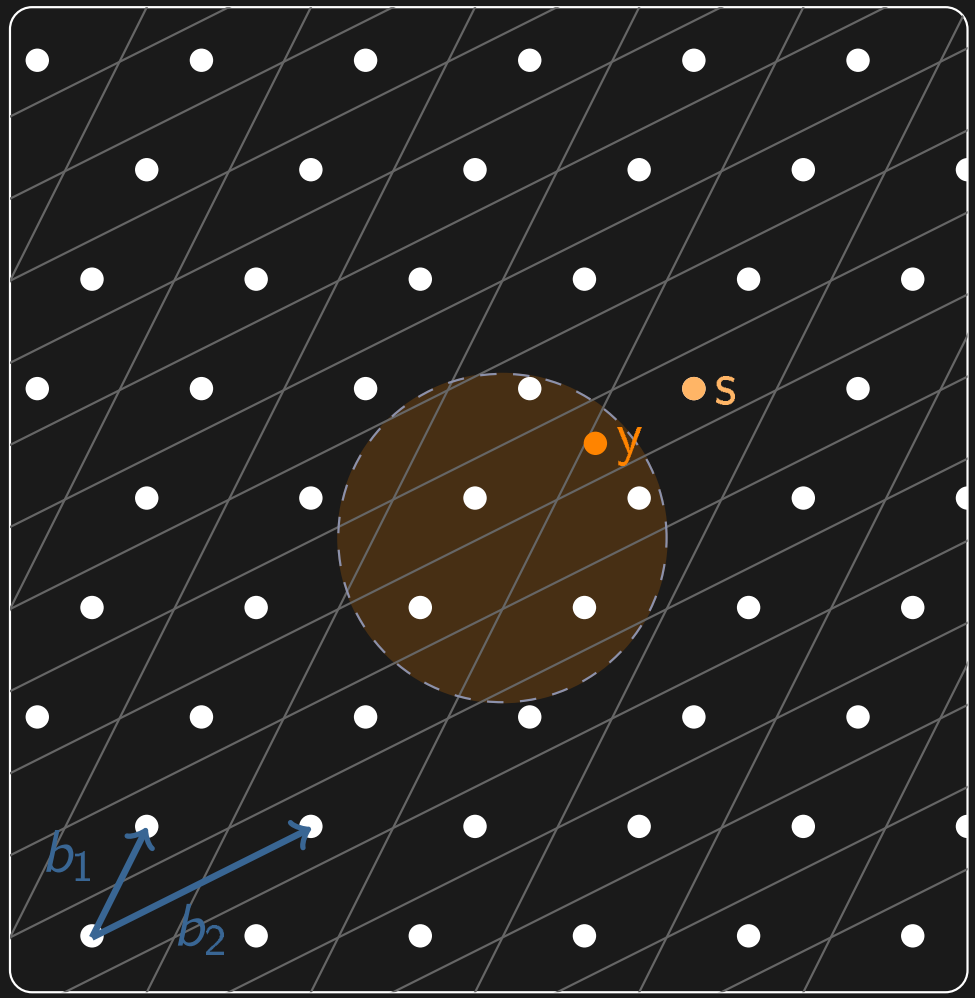


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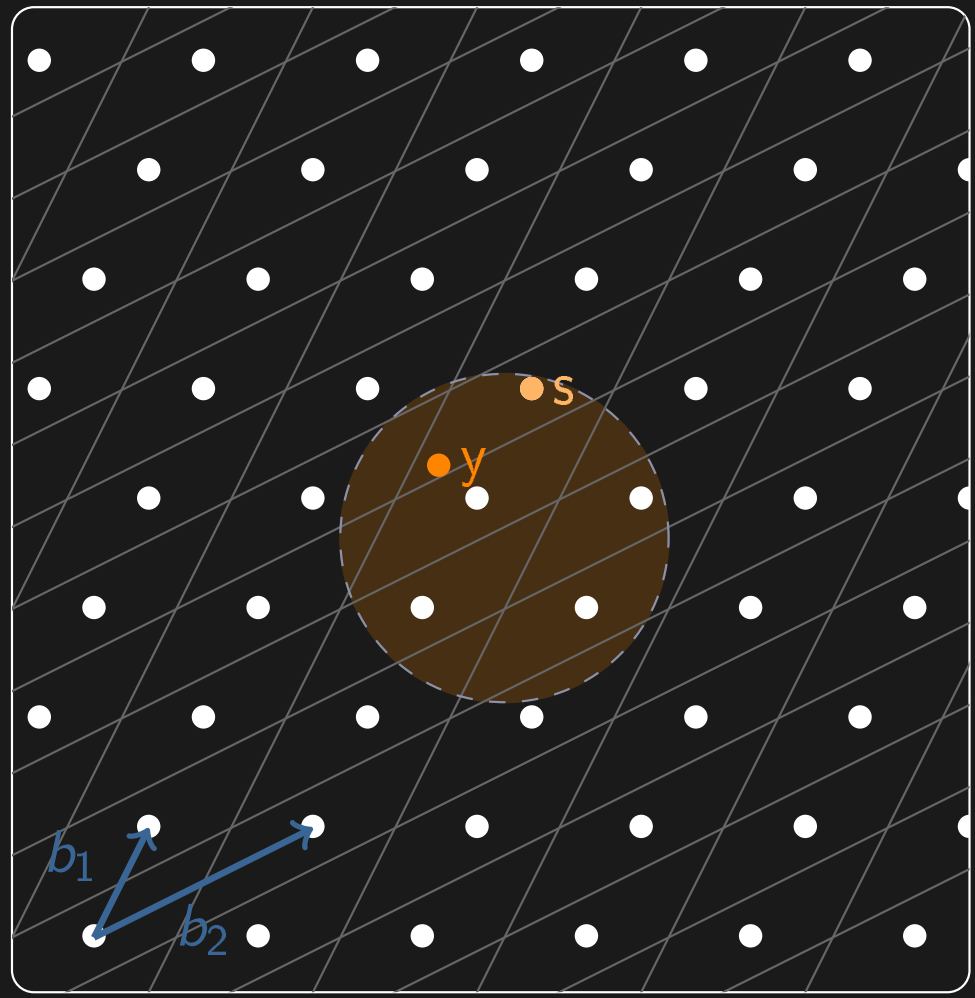


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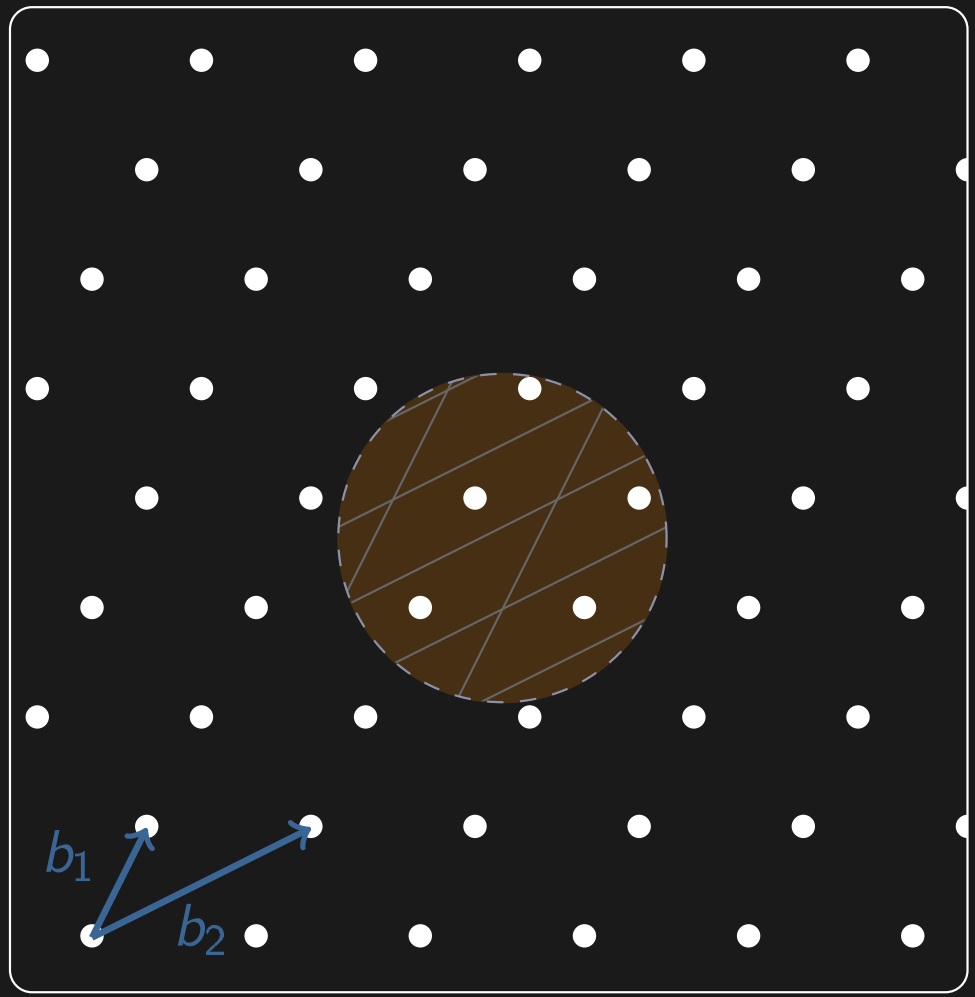


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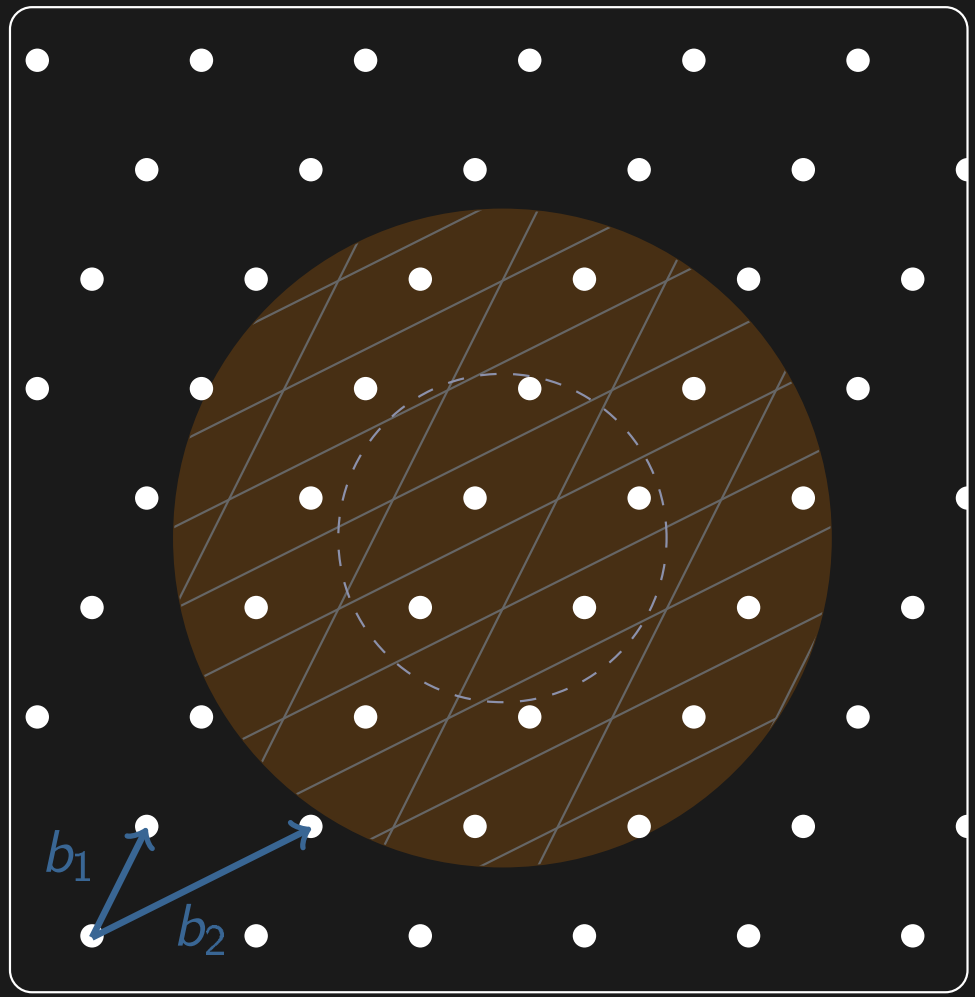


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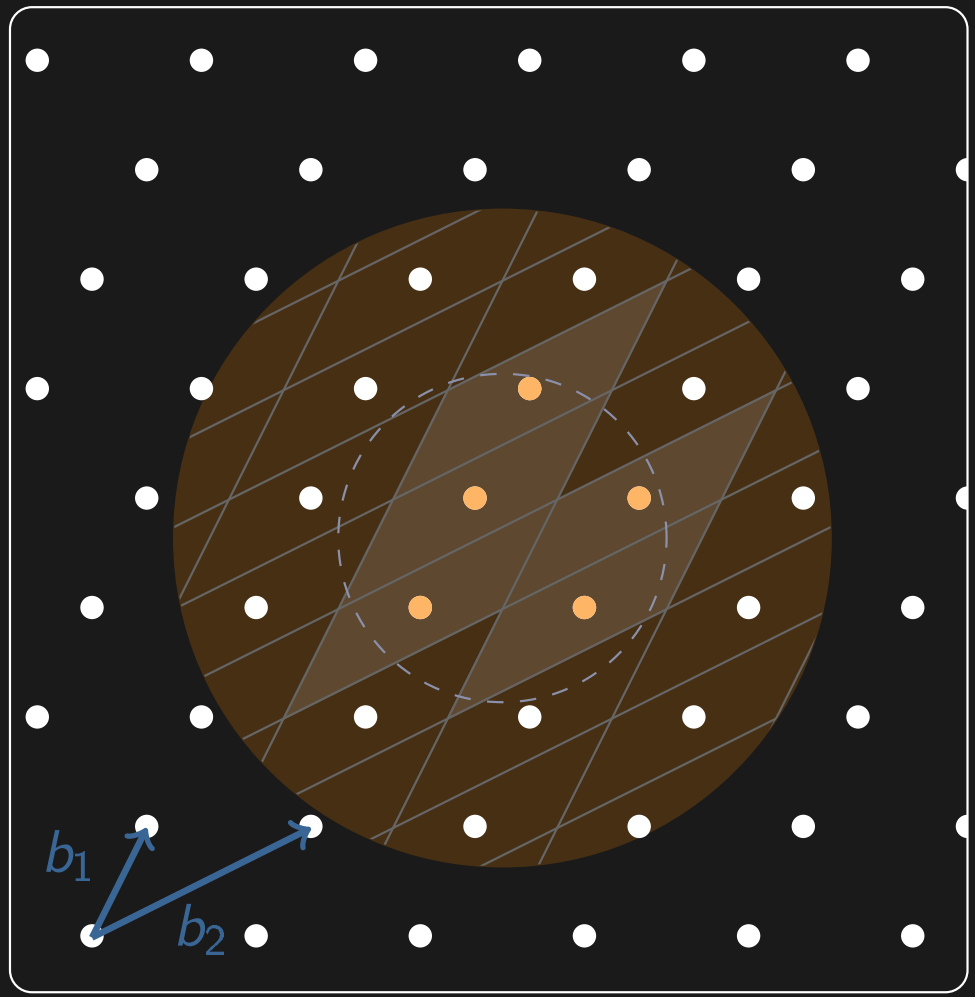


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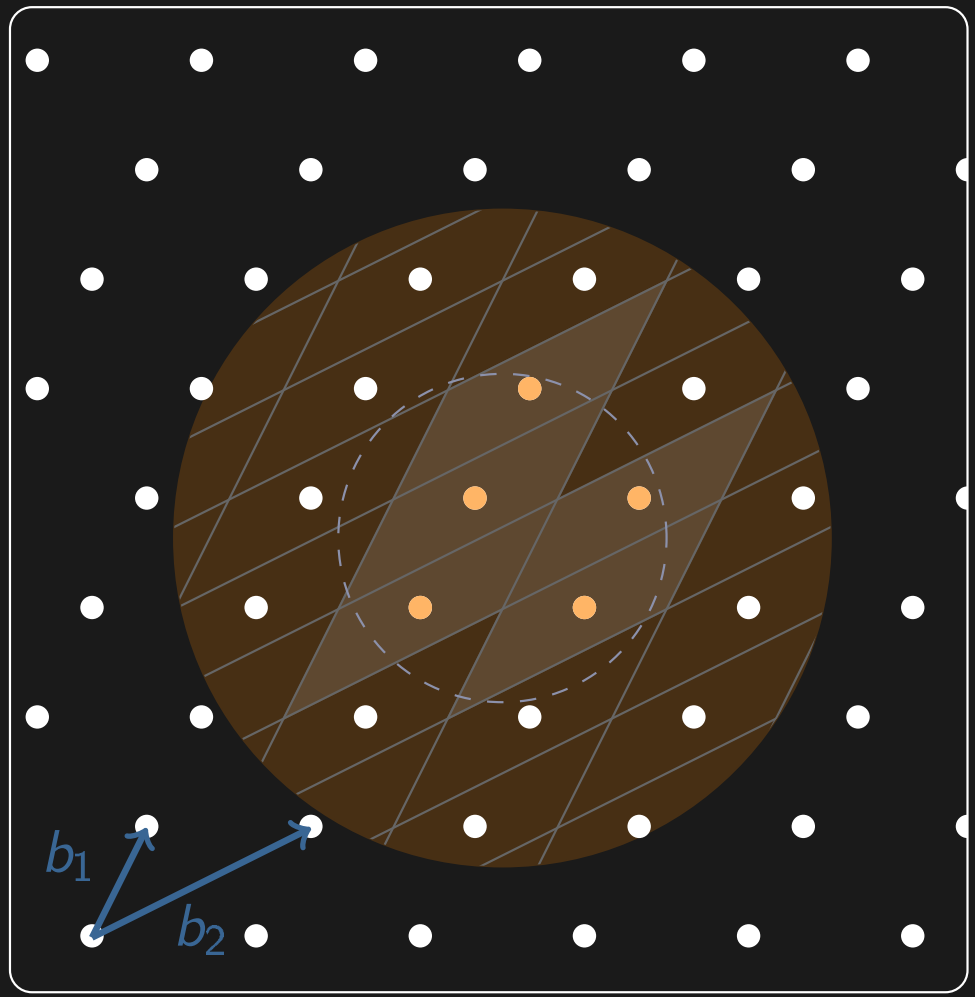
polynomial time if

$r \geq 2n^2 \cdot \max_i \|b_i\|$

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# Contradicting knowledge SIS (adapted from [WW23])

## Reminder: knowledge SIS

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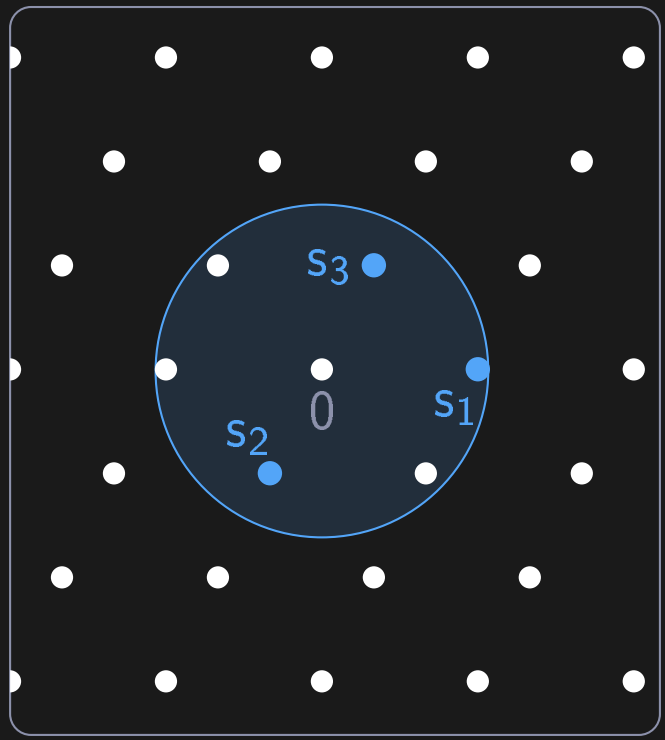
the only way to find $s^* \in \mathcal{L}$ short, is by taking small combinations of the $s_i$'s




[WW23] Wee and Wu. Lattice-based functional commitments: Fast verification and cryptanalysis. Asiacrypt

# Contradicting knowledge SIS (adapted from [WW23])

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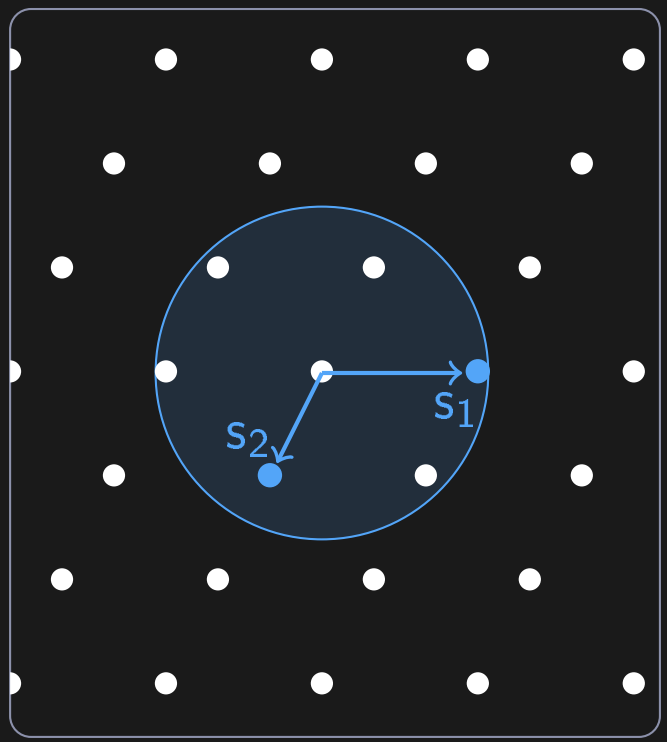


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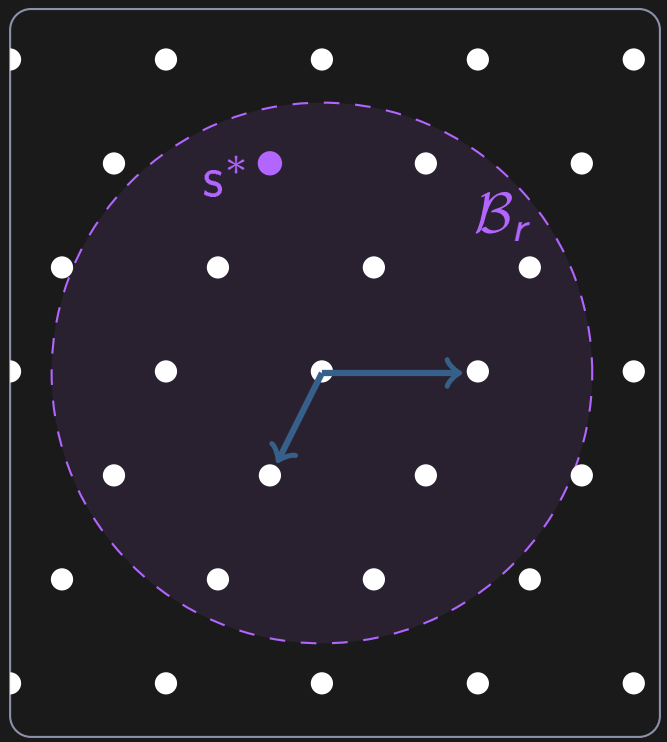


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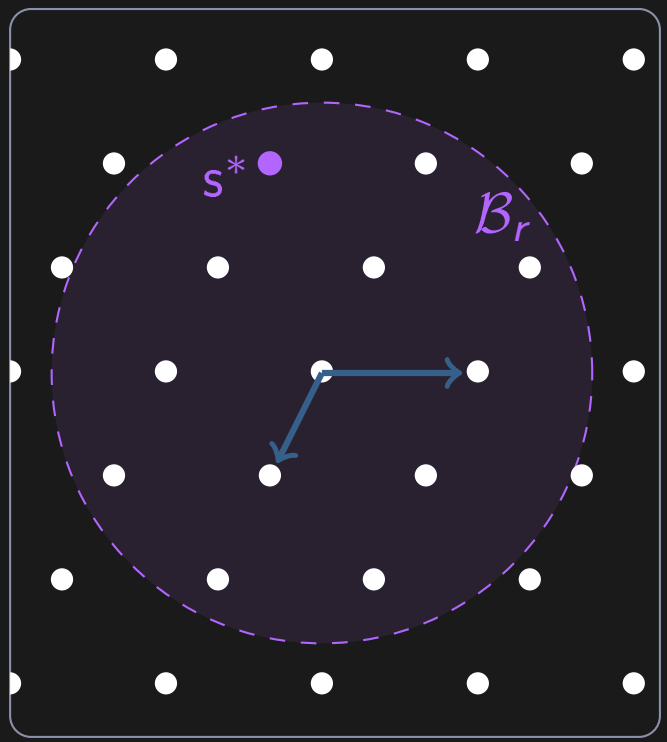


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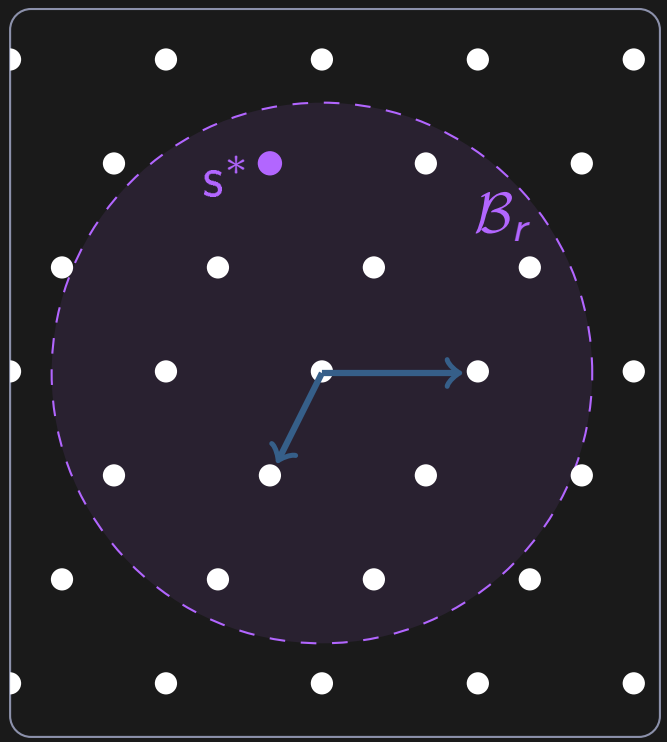


### Properties

- $\rightsquigarrow$ $s^* \in \mathcal{L}$ is somewhat short
- $\rightsquigarrow$ $s^*$ is not a small combination of $s_1, \ldots, s_n$ with high probability
  (it might be a small combination of $s_1, \ldots, s_k$)

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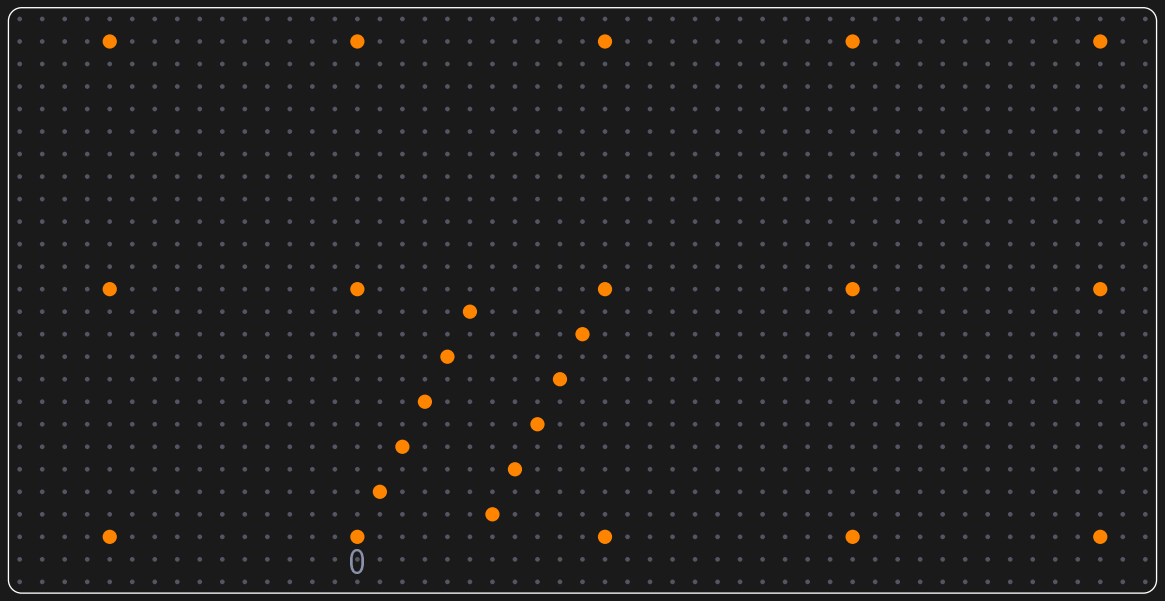
# The NTRU problem



# NTRU, matrix formalism [HPS98]

## NTRU problem

**Parameters:** $q$ prime, $B \ll \sqrt{q}$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

Input: $h \in \mathbb{Z}_q$

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**NTRU assumption**

Solving the NTRU problem is hard

---

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NTRU problem (matrix formalism): given $h \in \mathbb{Z}_q$ (for which a solution exists), find $(f, g)$ with $0 < |f|, |g| \leq B$ and $h = f \cdot g^{-1} \bmod q$

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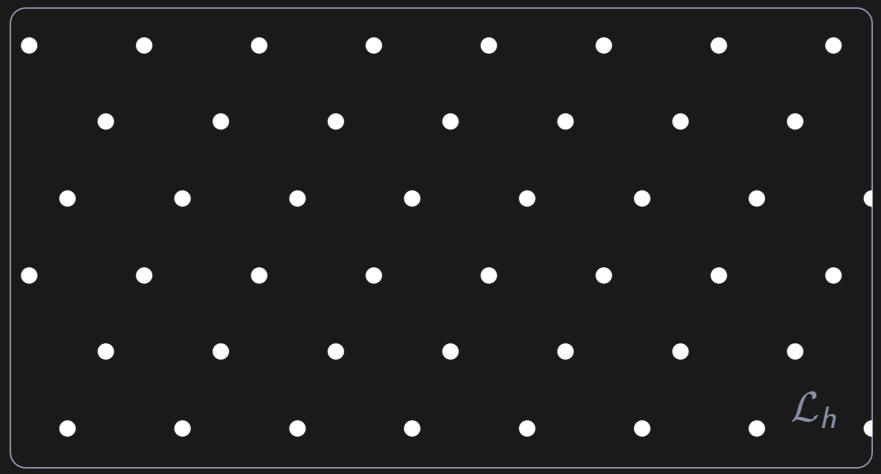

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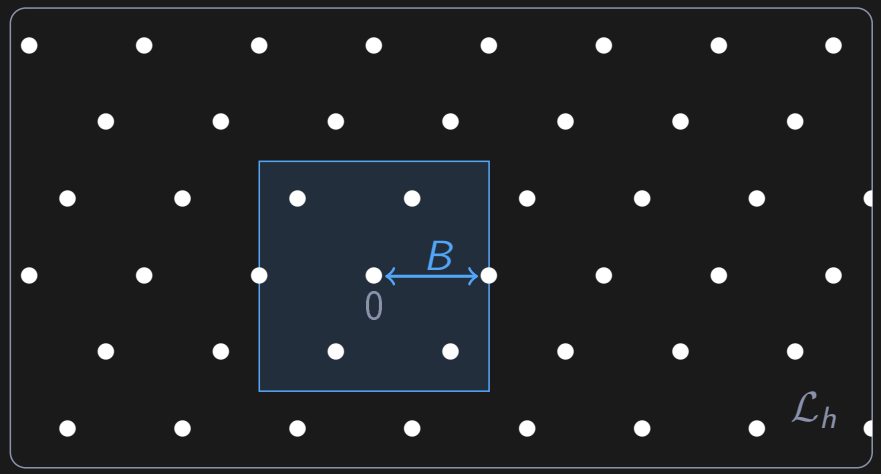


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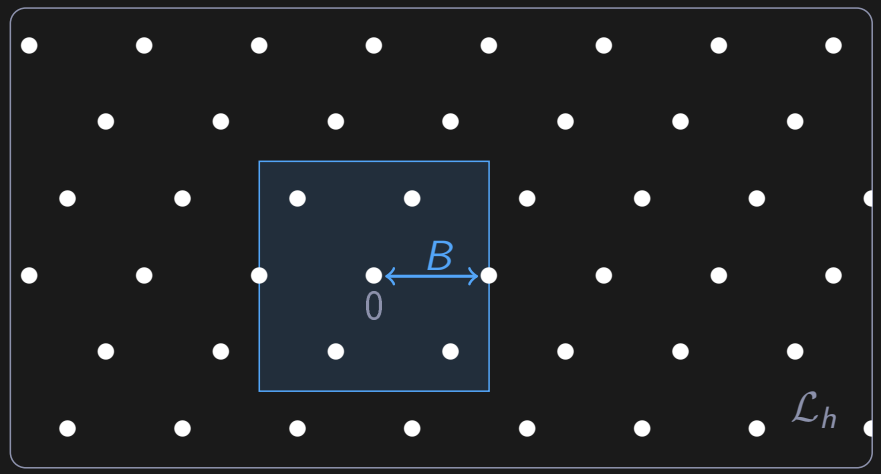


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solving NTRU $\iff$ finding a short non-zero vector in $\mathcal{L}_h$

# Summing up

**Reminder:** SVP = short vector problem

We have seen


(if you can solve SVP, you can also solve NTRU)

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How about the other direction? ...


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How about the other direction? . . .


(if you can solve NTRU, can you solve SVP?)

. . . we don't know (but it seems unlikely)

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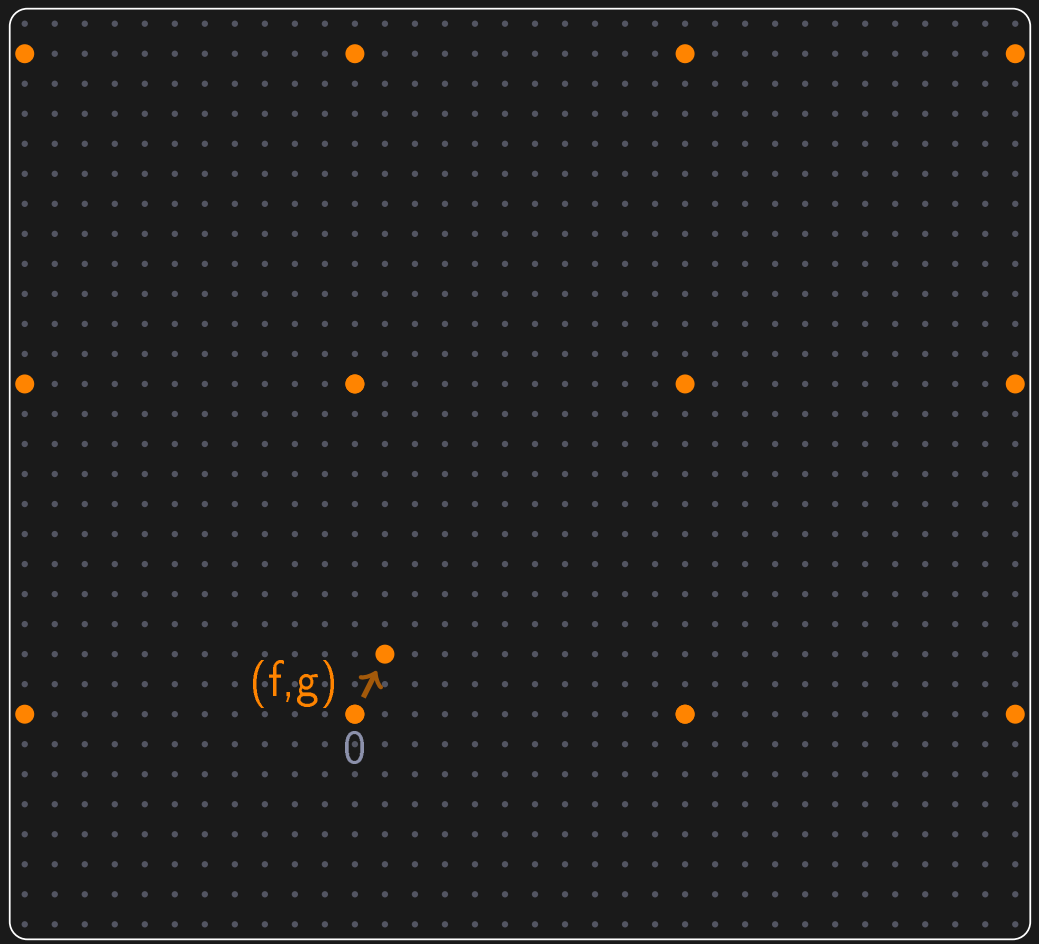


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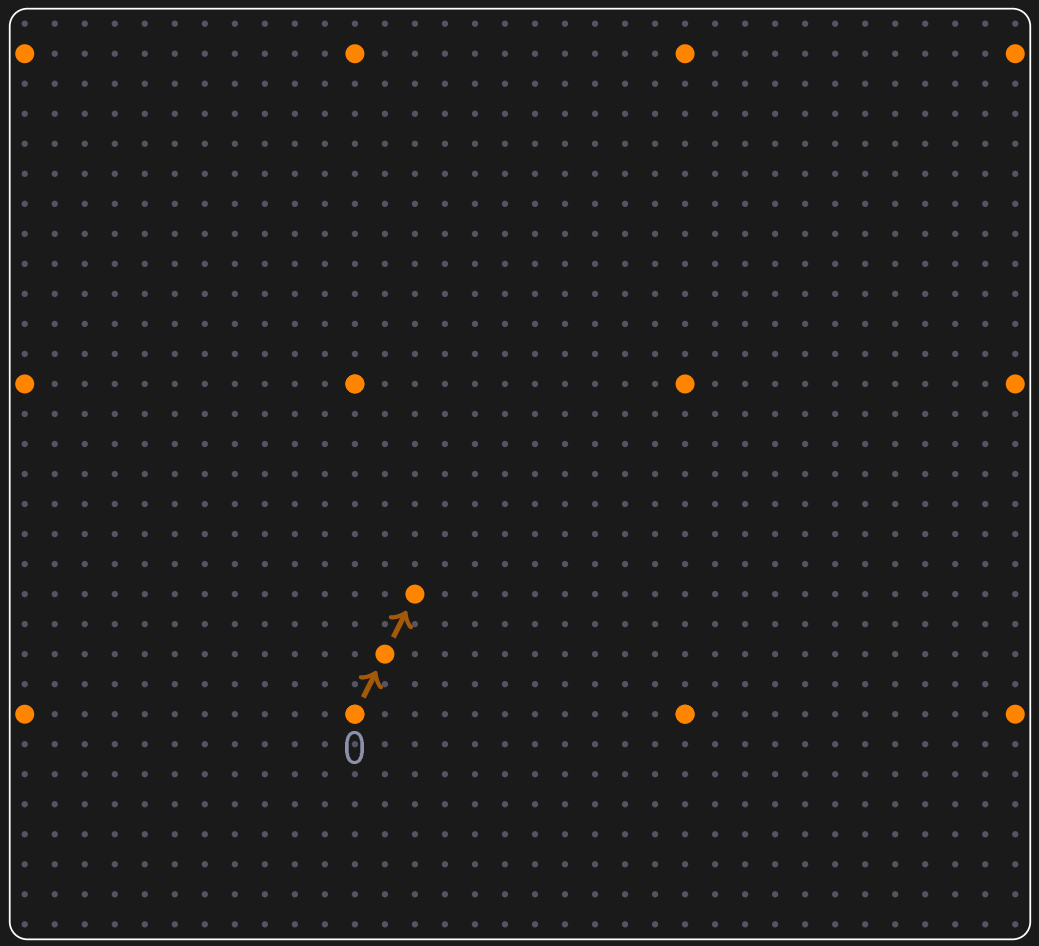


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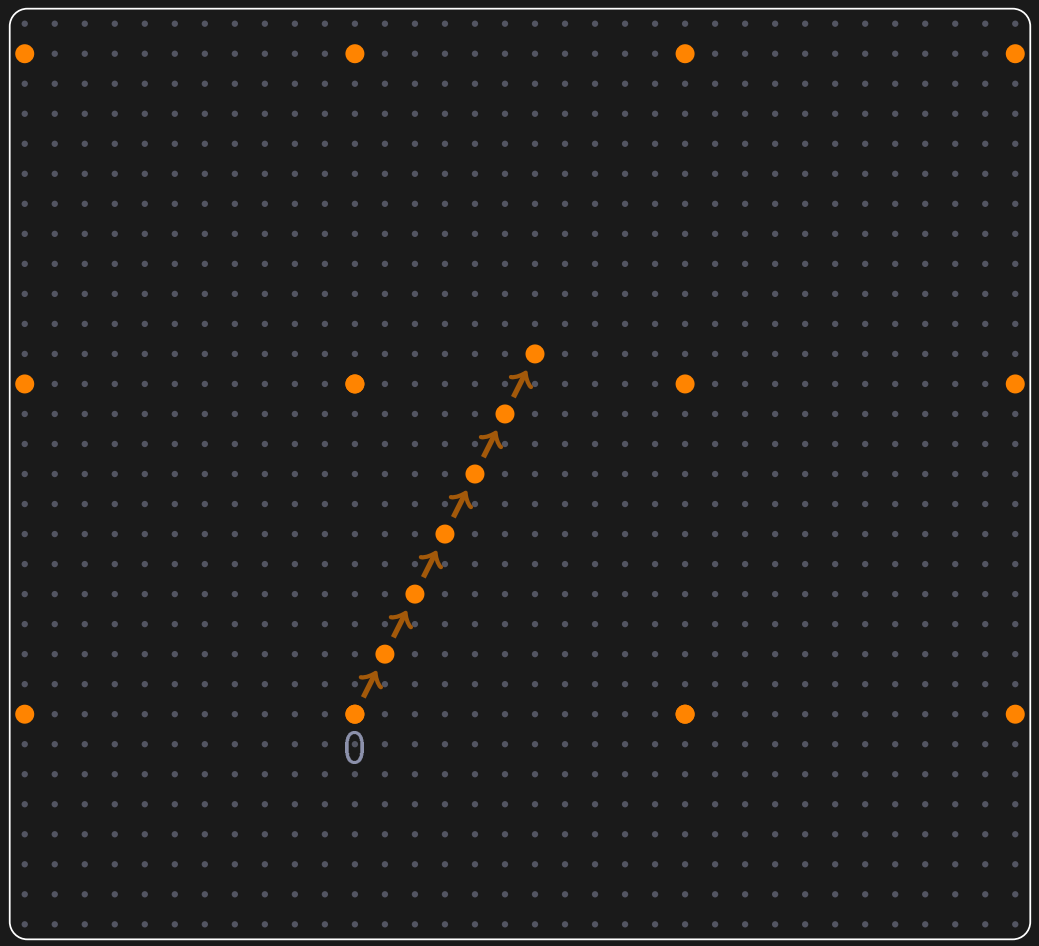


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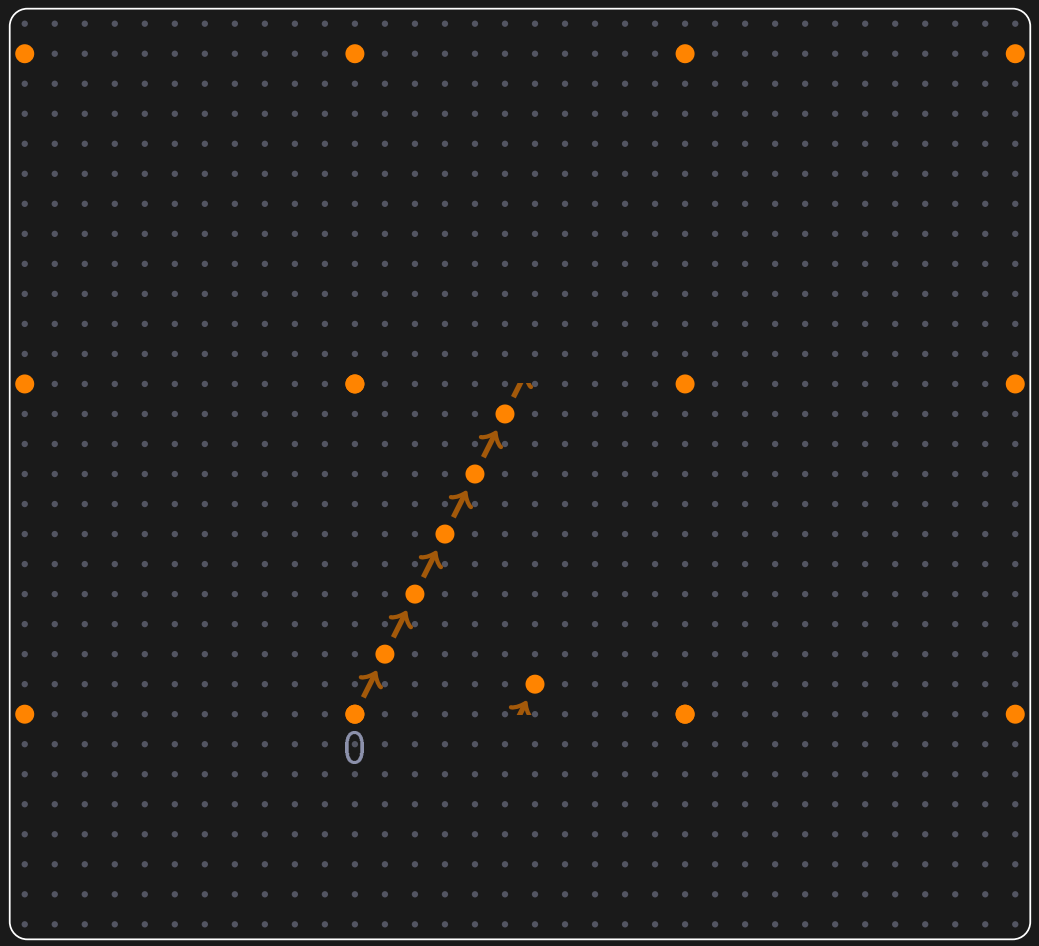


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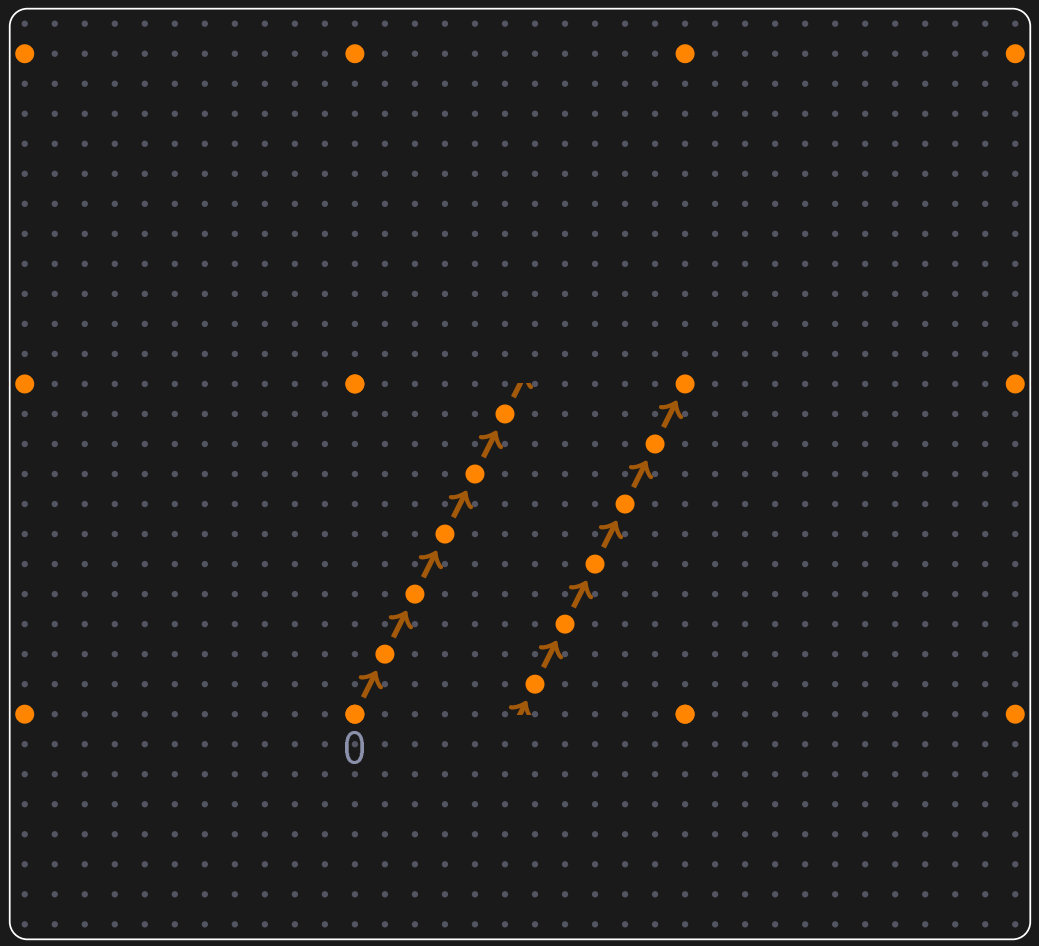


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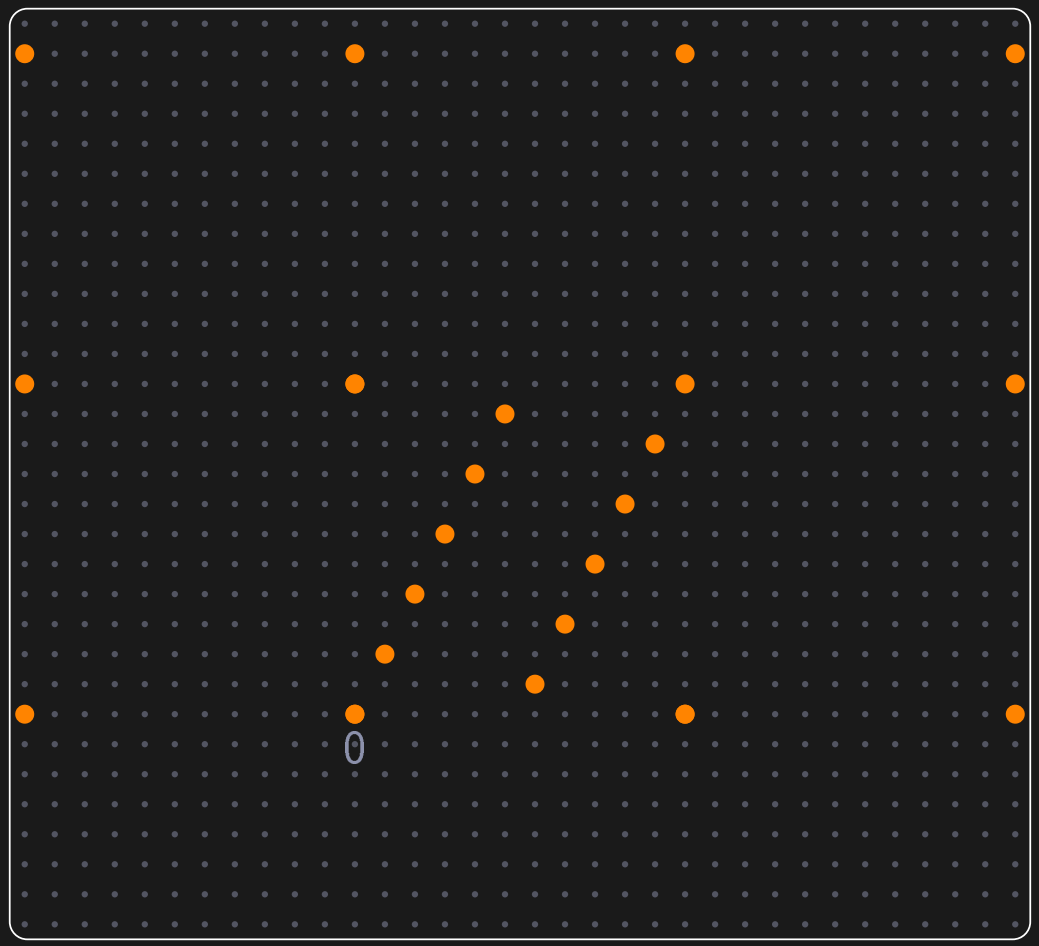


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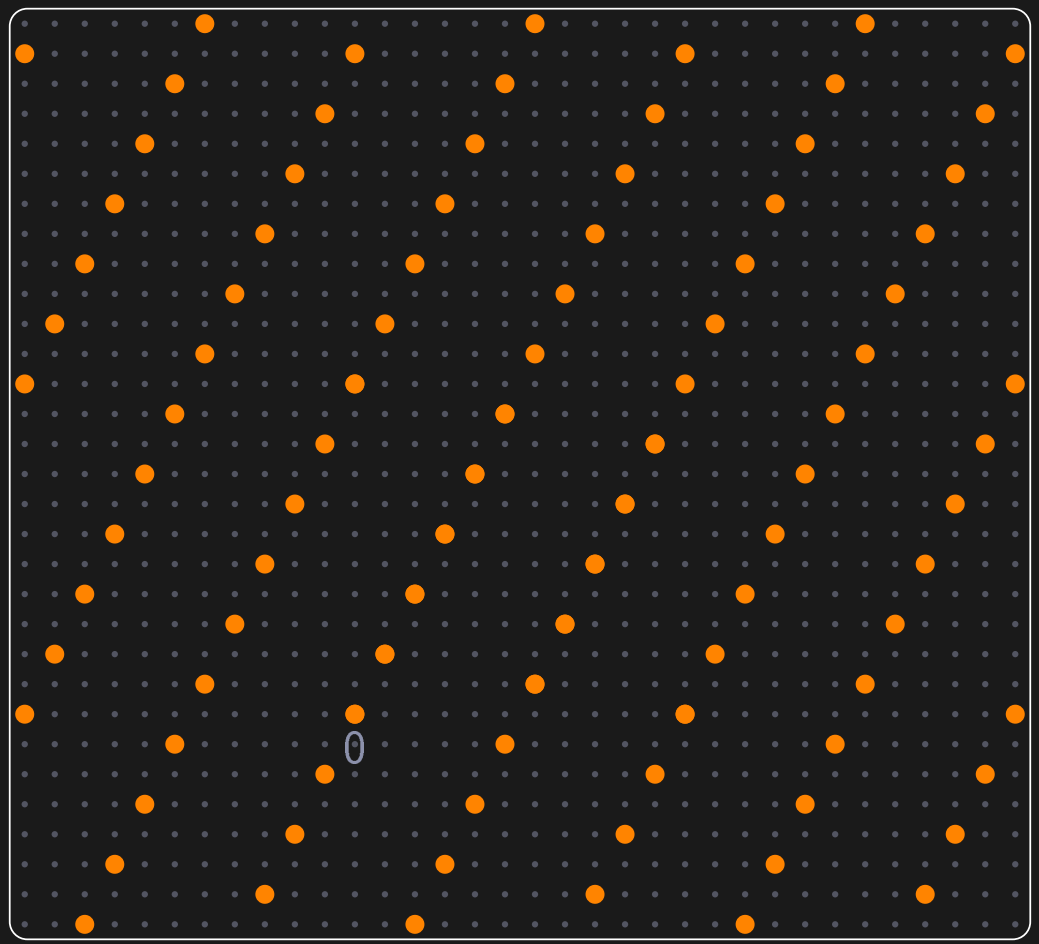


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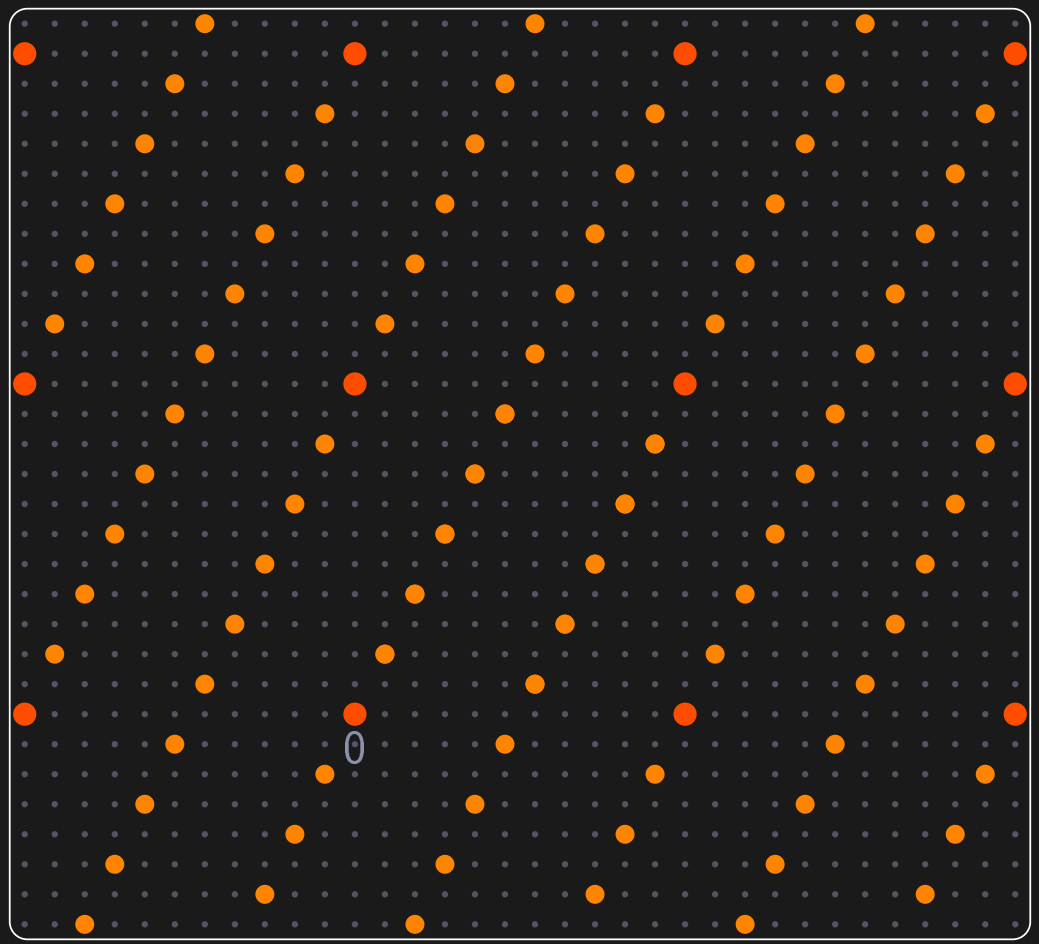


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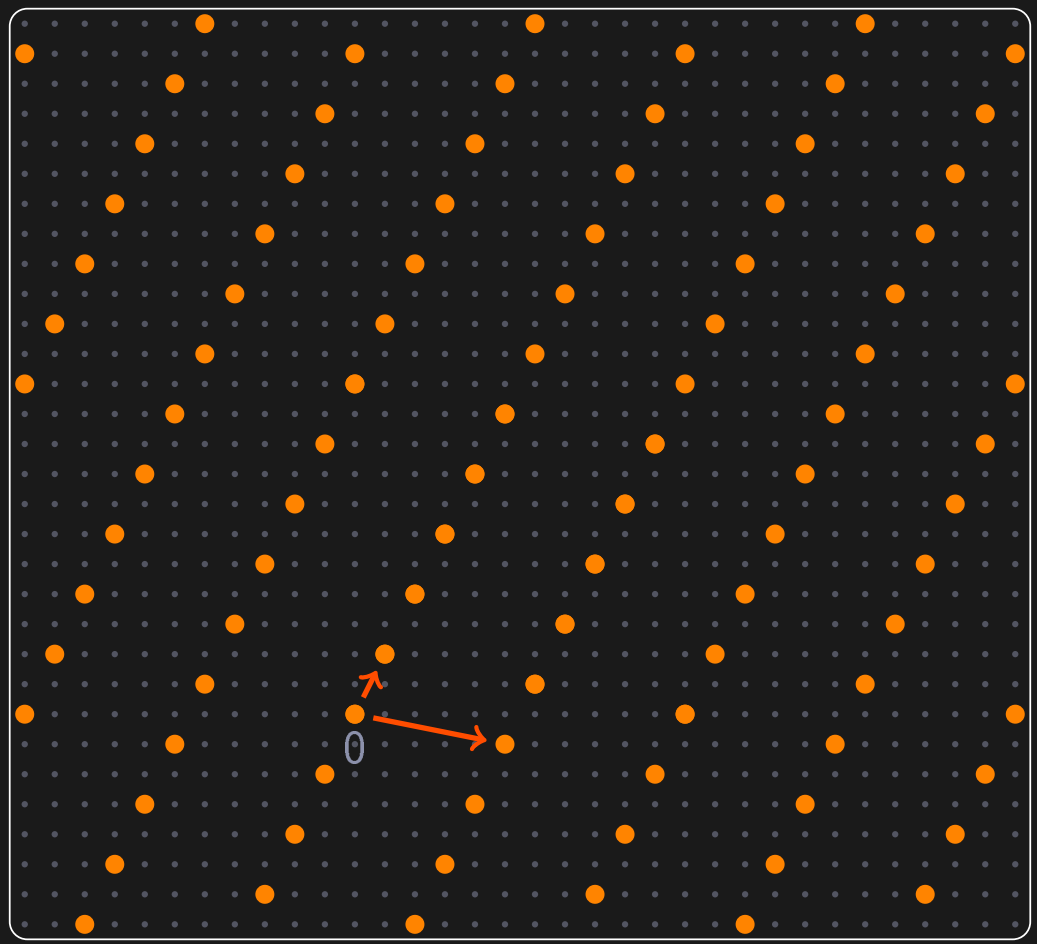


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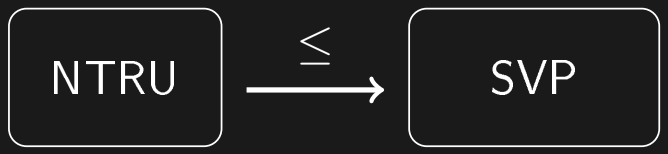
- $\mathcal{L}_h$ is unbalanced
  (its shortest basis is unbalanced)

# Back to comparisons

**Reminder:** SVP = short vector problem

We have seen

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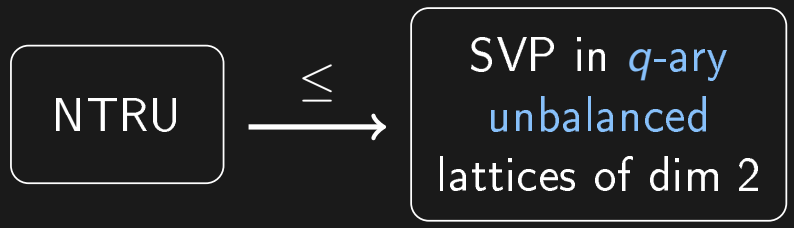


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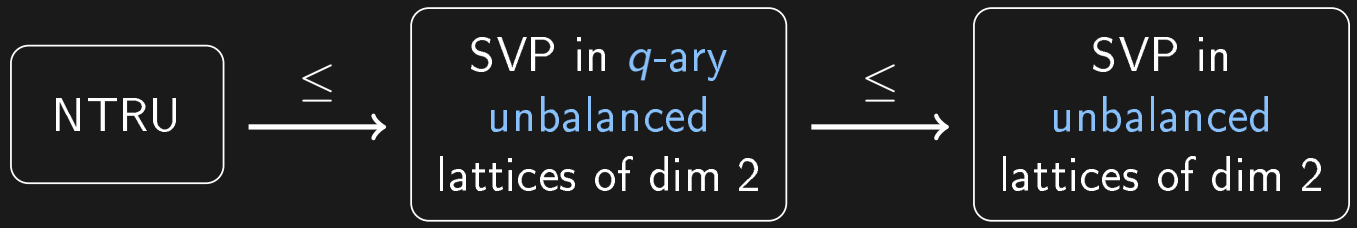


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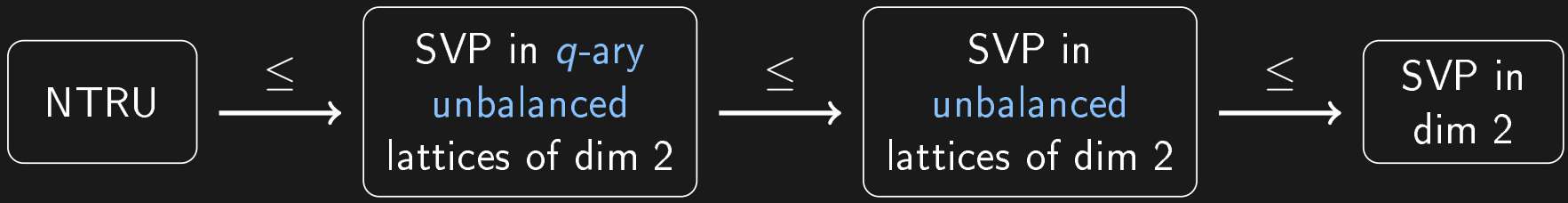


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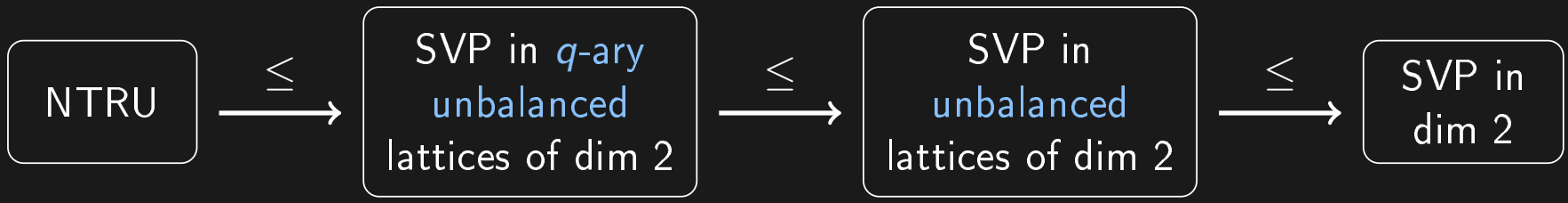


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## Now we can partially prove the other direction

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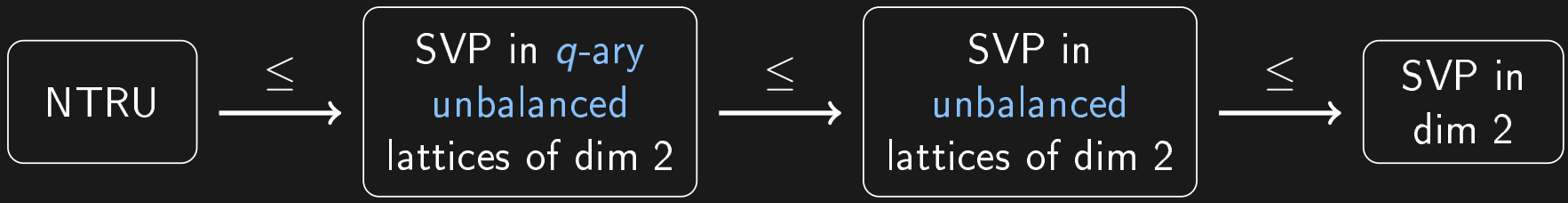
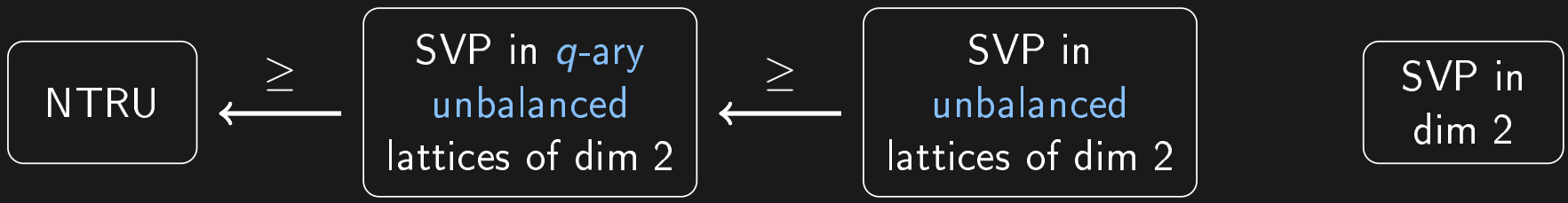



($A \geq B \iff$ if you can solve $A$, you can also solve $B$)

## Now we can partially prove the other direction



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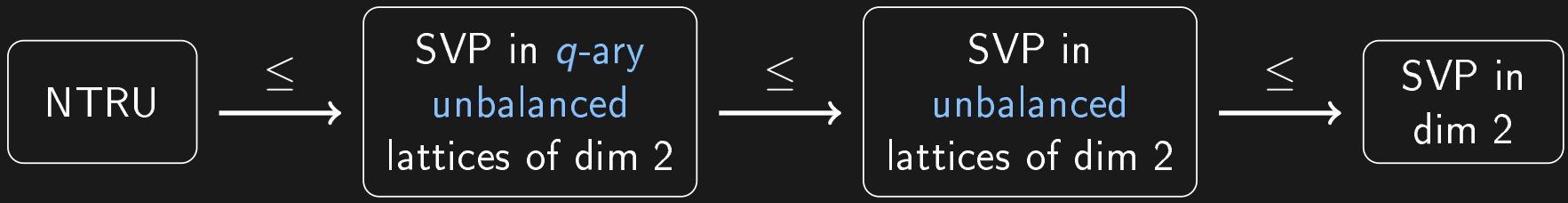


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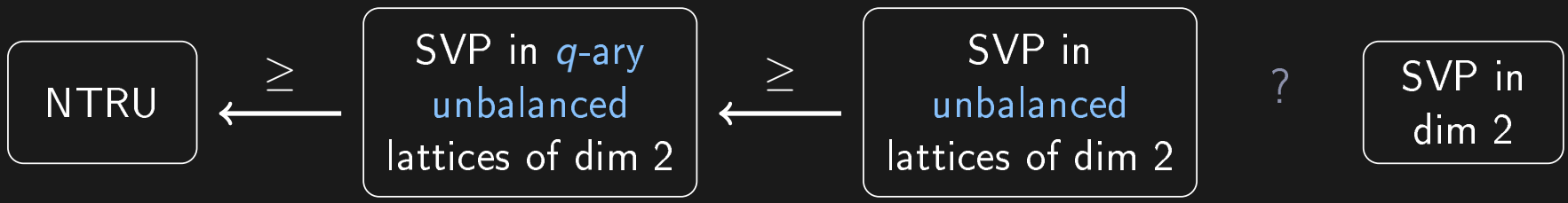

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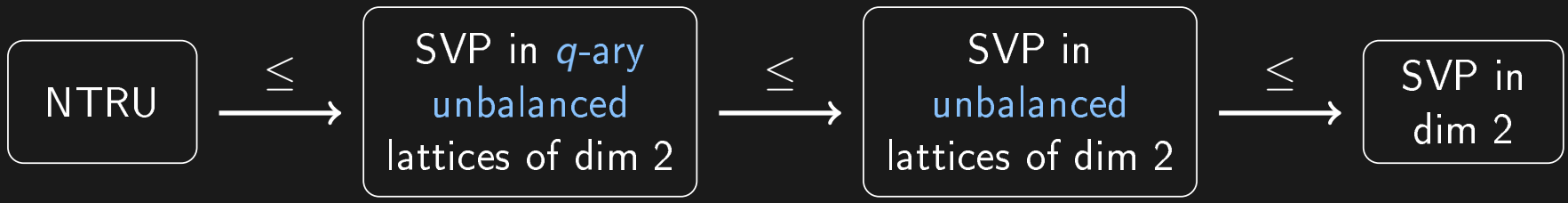


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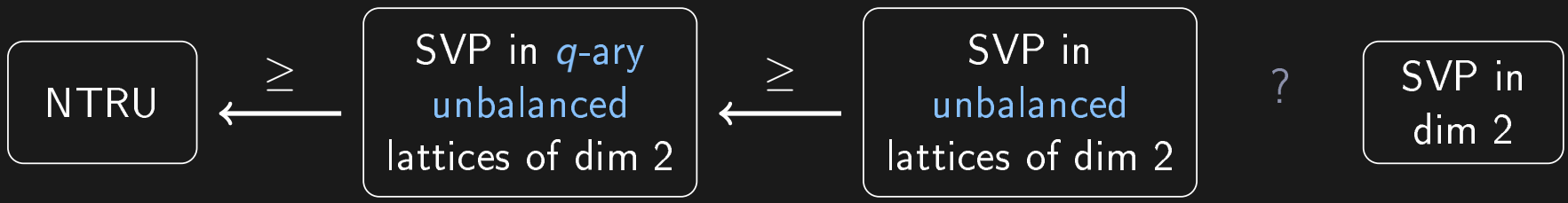


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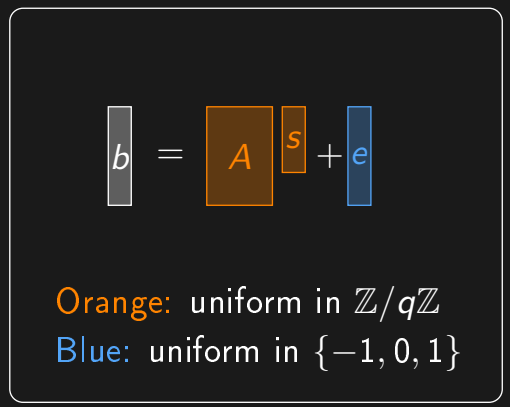
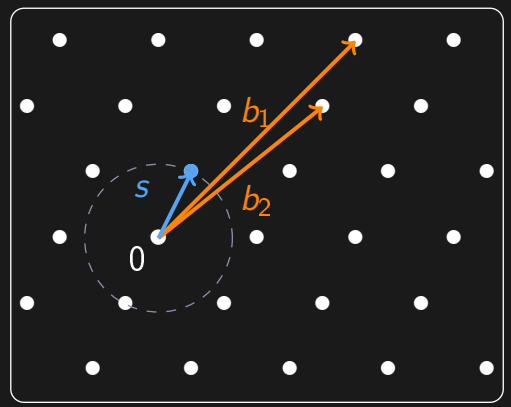
- $\mathcal{L}_h = \left\{(u, v) \in (\mathbb{Z}[X]/(X^d + 1))^2 \cong \mathbb{Z}^{2d} \,\middle|\, v \cdot h = u \bmod q\right\}$
  - lattice of dimension 2d (or module lattice of rank 2)

NTRU $\approx$ SVP in unbalanced module lattice of rank 2

# Conclusion


b = A s + e
Orange: uniform in $\mathbb{Z}/q\mathbb{Z}$
Blue: uniform in $\{-1, 0, 1\}$


$b_1$
$b_2$
s
0

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Thank you