Approx-SVP in Ideal Lattices with Pre-Processing

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ENS de Lyon

RISC and Prometheus seminar
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https://eprint.iacr.org/2019/215.pdf

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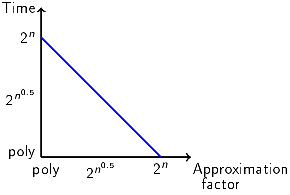
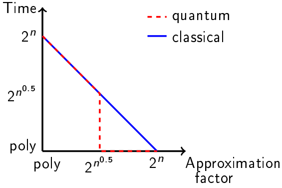
What is this talk about

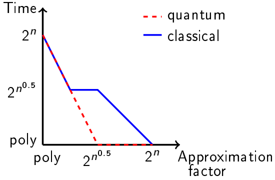
Time/Approximation factor trade-off for SVP in ideal lattices:



BKZ algorithm [Sch87]

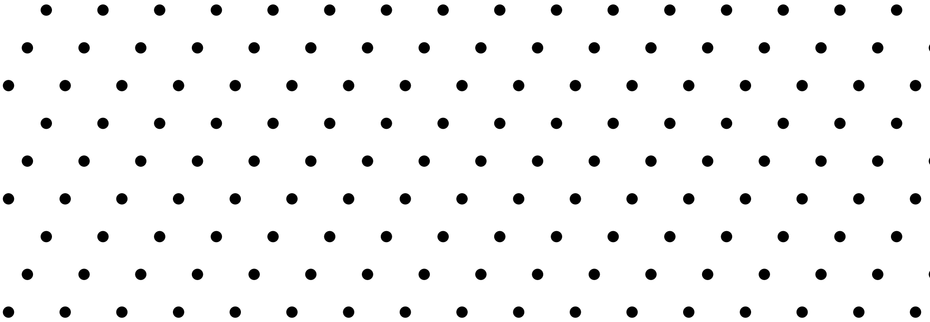


[CDW17]



This work
(with $2^{O(n)}$ pre-processing)

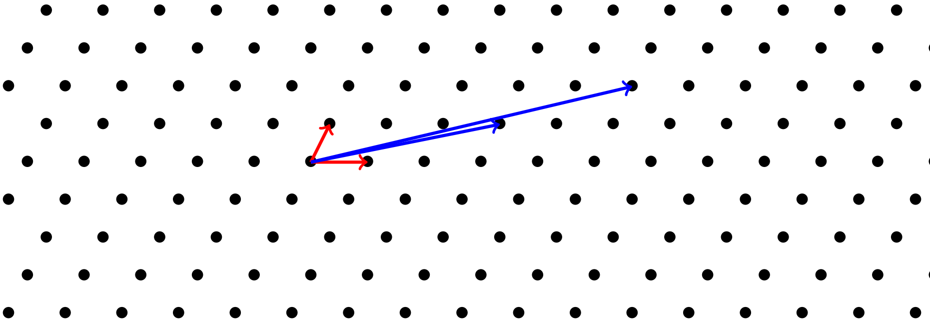
Lattices



Lattice

A lattice L is a discrete 'vector space' over $\mathbb{Z}$.

Lattices

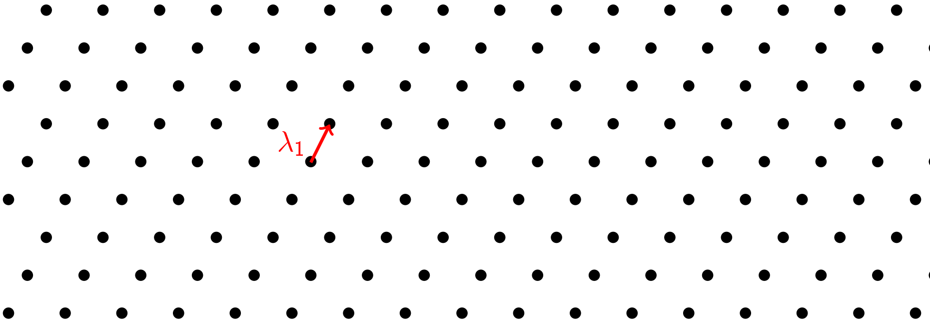


Lattice

A lattice L is a discrete 'vector space' over $\mathbb{Z}$.
A basis of L is an invertible matrix B such that $L = \{Bx \mid x \in \mathbb{Z}^n\}$.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 10 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

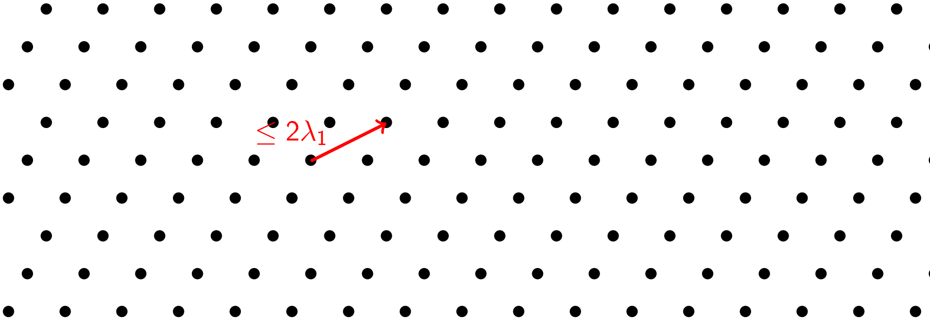
Lattices



Shortest Vector Problem (SVP)

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted λ_1.

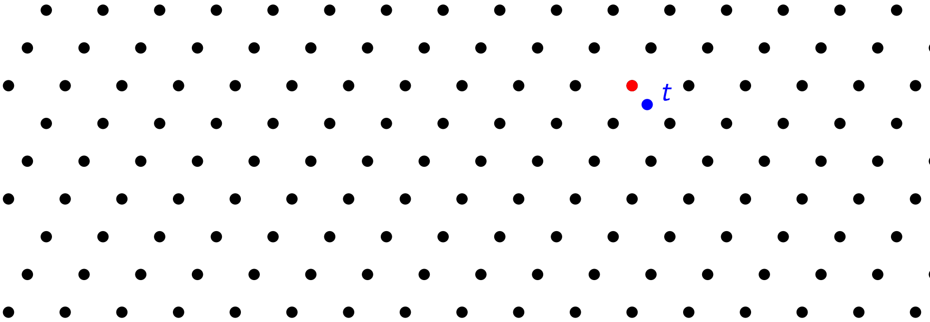
Lattices



Approximate Shortest Vector Problem (approx-SVP)

Find a short (in Euclidean norm) non-zero vector.
(e.g. of norm $\leq 2\lambda_1$).

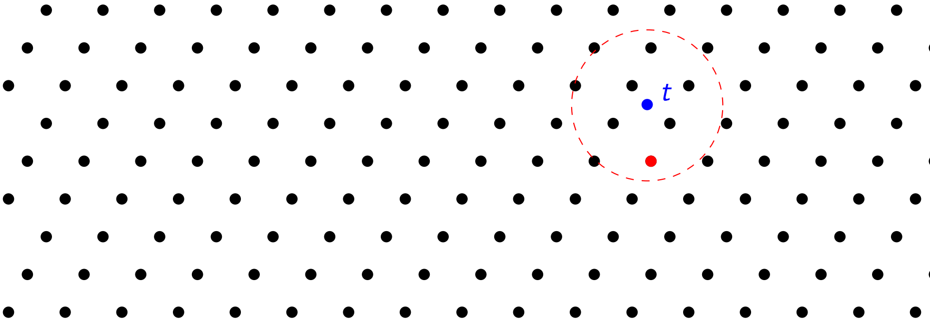
Lattices



Closest Vector Problem (CVP)

Given a target point t, find a point of the lattice closest to t.

Lattices



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Complexity of SVP/CVP

Applications

Approx-SVP and approx-CVP in generic lattices are conjectured to be hard to solve both quantumly and classically $\Rightarrow$ used in cryptography

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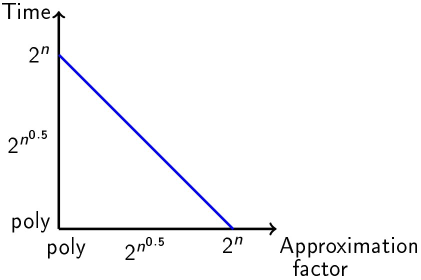
Best Time/Approx trade-off for generic lattices: BKZ algorithm [Sch87]



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Structured lattices

Improve efficiency of lattice-based crypto using structured lattices.
$\Rightarrow$ E.g. ideal lattices

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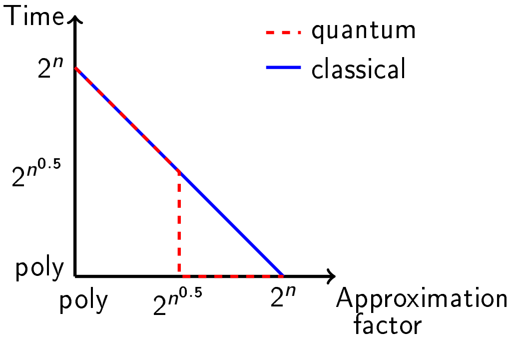
Is approx-SVP still hard when restricted to ideal lattices?

SVP in ideal lattices

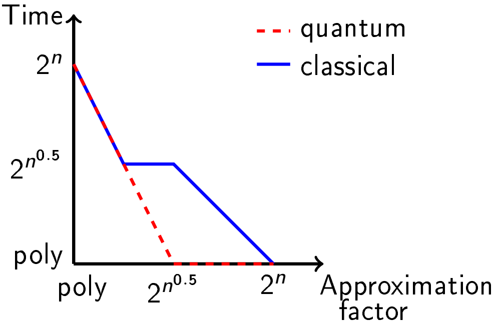
[CDW17]: Better than BKZ in the quantum setting



- Heuristic
- For prime power cyclotomic fields

[CDW17] R. Cramer, L. Ducas, B. Wesolowski. Short Stickelberger Class Relations and Application to Ideal-SVP, Eurocrypt.

This work



- Heuristic
- Pre-processing $2^{O(n)}$, independent of the choice of the ideal (non-uniform algorithm).

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  - very few schemes based in ideal-SVP [Gen09,GGH13]

$$\boxed{\text{schemes}} \longrightarrow \boxed{\text{RLWE}} \longrightarrow \boxed{\text{ideal SVP}}$$

[Gen09] C. Gentry. Fully homomorphic encryption using ideal lattices, STOC.

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$$\begin{tikzcd} \boxed{\text{schemes}} \arrow[r] & \boxed{\text{RLWE}} \arrow[r, shift left] \arrow[d, leftrightarrow] & \boxed{\text{ideal SVP}} \arrow[l, shift left, "?"] \\ & \boxed{\text{module SIVP}} & \end{tikzcd}$$

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Outline of the talk

1. Definitions and objective
2. The CDPR algorithm
3. This work
4. Extension: “Euclidean division” over R

First definitions

Notation

$R = \mathbb{Z}[X]/(X^n + 1)$ for $n = 2^k$ (for simplicity)

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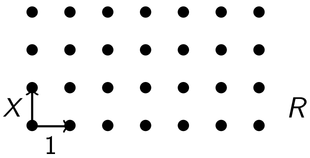
- Units: $R^\times = \{a \in R \mid \exists b \in R, ab = 1\}$
  - e.g. $\mathbb{Z}^\times = \{-1, 1\}$
- Principal ideals: $\langle g \rangle = \{gr \mid r \in R\}$ (i.e. all multiples of g)
  - e.g. $\langle 2 \rangle = \{\text{even numbers}\}$ in $\mathbb{Z}$
  - g is called a generator of $\langle g \rangle$
  - The generators of $\langle g \rangle$ are exactly the ug for $u \in R^\times$

Why is $\langle g \rangle$ a lattice?

$R \simeq \mathbb{Z}^n$

$$R = \mathbb{Z}[X]/(X^n + 1) \to \mathbb{Z}^n$$
$$r = r_0 + r_1 X + \cdots + r_{n-1} X^{n-1} \mapsto (r_0, r_1, \dots, r_{n-1})$$



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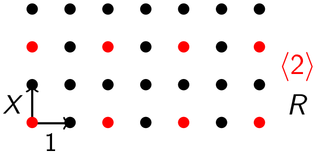
$\langle g \rangle \subseteq R \simeq \mathbb{Z}^n$ + stable by '+' and '-' $\Rightarrow$ lattice



Objective of this talk

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Given a basis of a principal ideal $\langle g \rangle$ and $\alpha \in (0,1]$,
Find $r \in \langle g \rangle$ such that $\|r\| \leq 2^{\widetilde{O}(n^\alpha)} \cdot \lambda_1$.

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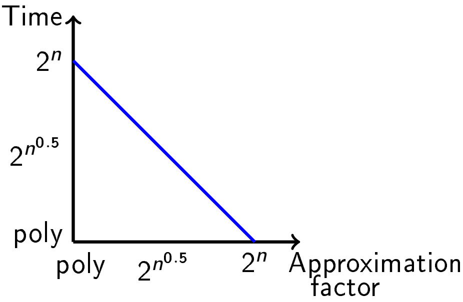
BKZ algorithm can do it in time $2^{O(n^{1-\alpha})}$, can we do better?



The CDPR algorithm

Main idea of the CDPR algorithm (on an idea of [CGS14])

Idea

Maybe g is a somehow small element of $\langle g \rangle$

[CDPR16] R. Cramer, L. Ducas, C. Peikert and O. Regev. Recovering Short Generators of Principal Ideals in Cyclotomic Rings, Eurocrypt.

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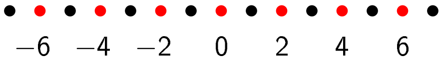

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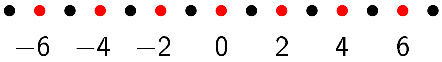

For larger n: one of the generators is somehow small

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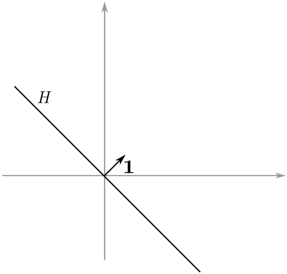
The Log space

Log : $R \to \mathbb{R}^n$ (somehow generalising log to R)

Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^{\perp}$.



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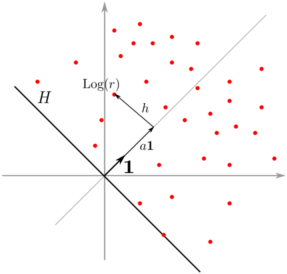

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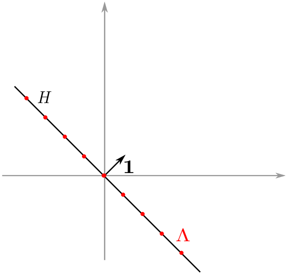

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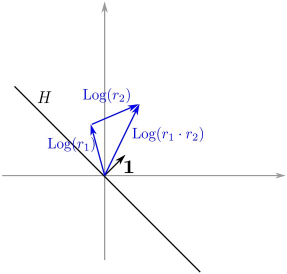

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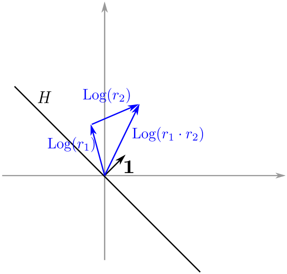
- $\text{Log}(r_1 \cdot r_2) = \text{Log}(r_1) + \text{Log}(r_2)$
- $\|r\| \simeq 2^{\|\text{Log}\, r\|_\infty}$



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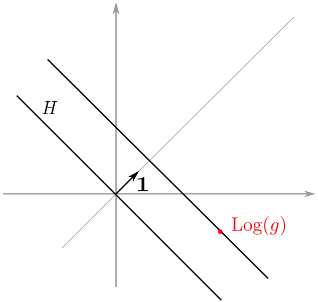
What does $\mathrm{Log}\langle g \rangle$ look like?



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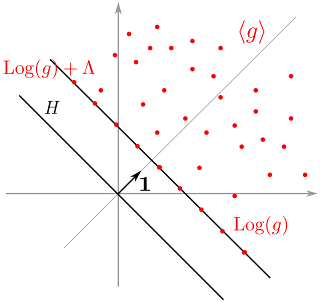

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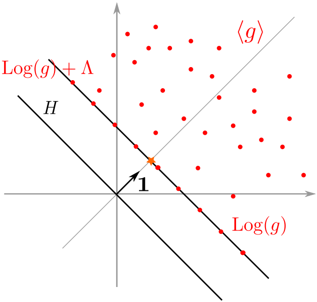

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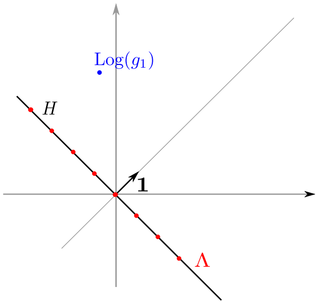
The CDPR Algorithm:

- Find a generator g_1 of $\langle g \rangle$.
  - [BS16]: quantum time $\mathrm{poly}(n)$
  - [BEFGK17]: classical time $2^{\widetilde{O}(\sqrt{n})}$



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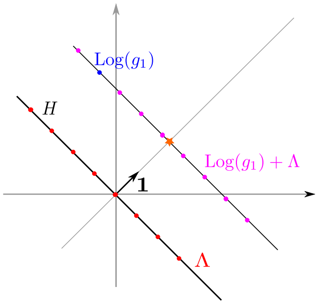

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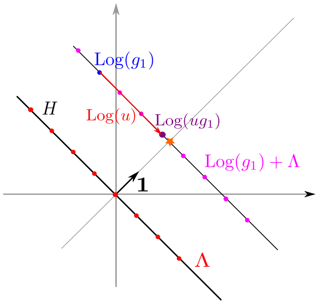

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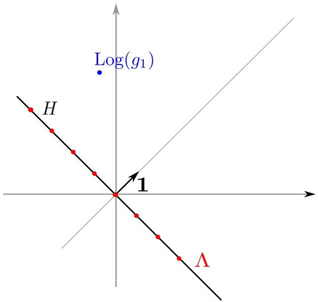

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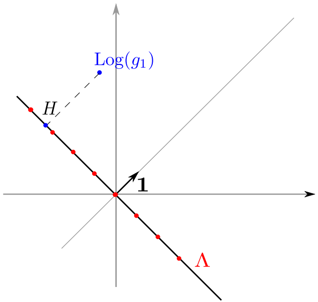

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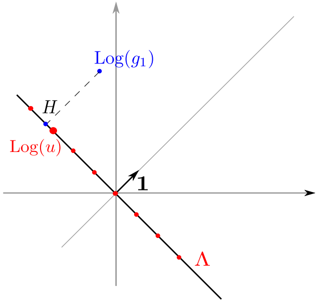

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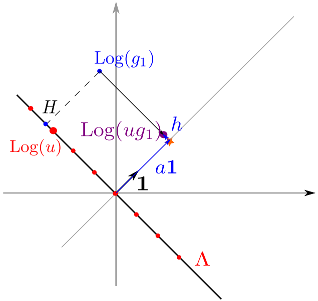

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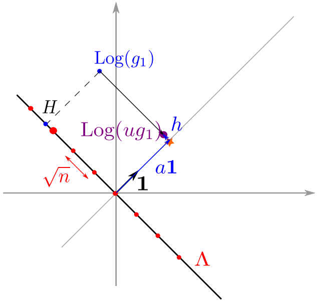
- Solve CVP in Λ
  - Good basis of Λ
    $\Rightarrow$ CVP in poly time
    $\Rightarrow \|h\| \leq \widetilde{O}(\sqrt{n})$



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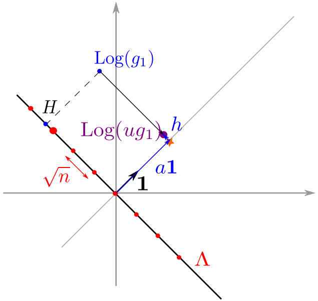
  - Good basis of Λ
    $\Rightarrow$ CVP in poly time
    $\Rightarrow \|h\| \leq \widetilde{O}(\sqrt{n})$



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The CDPR algorithm

The CDPR Algorithm:

- Find a generator g_1 of $\langle g \rangle$.
  - [BS16]: quantum time $\mathrm{poly}(n)$
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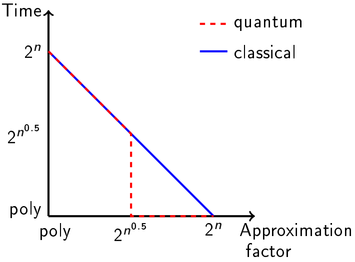

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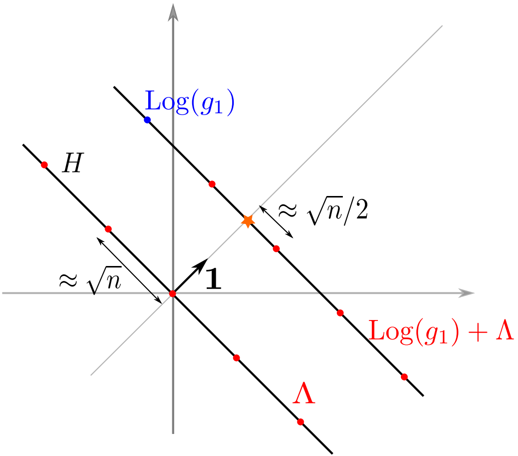
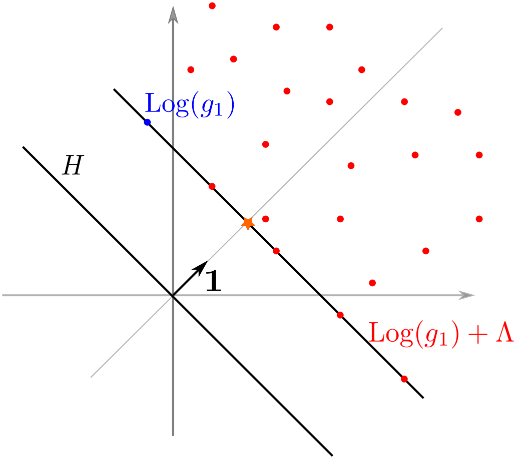
This work

Idea



$\mathrm{Log}(g_1)$
H
$\approx \sqrt{n}/2$
$\approx \sqrt{n}$
$\mathbf{1}$
$\mathrm{Log}(g_1) + \Lambda$
Λ

Idea



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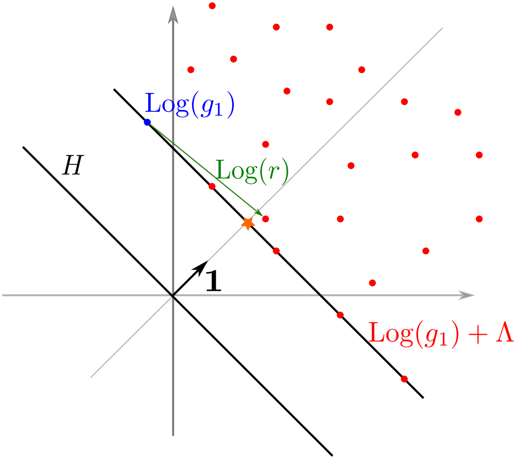

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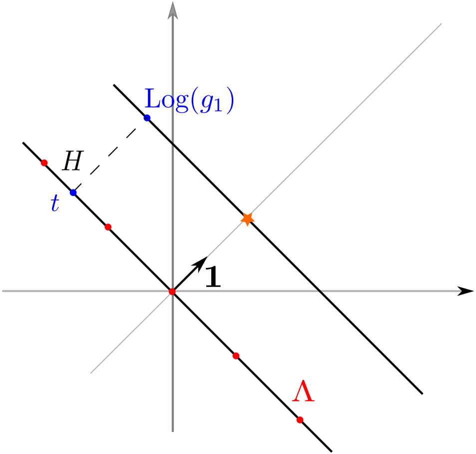

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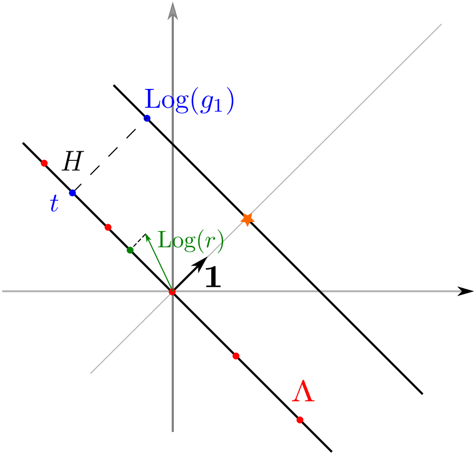

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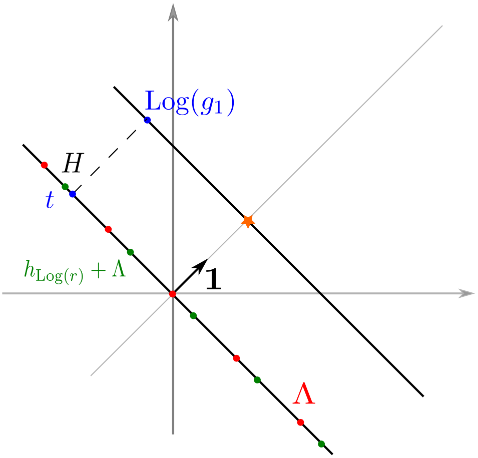

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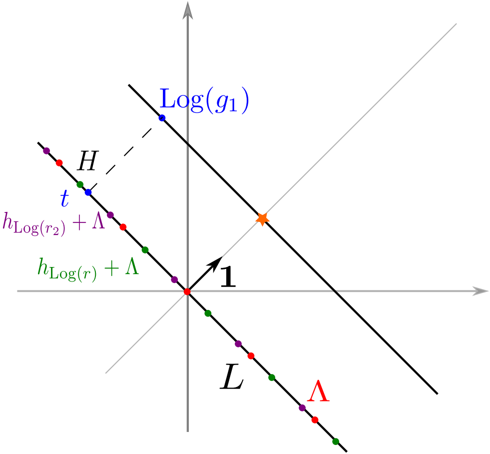

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How to solve CVP in L?

CDPR	This work
Good basis of Λ	No good basis of L known

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[DLW19,Ste19]:

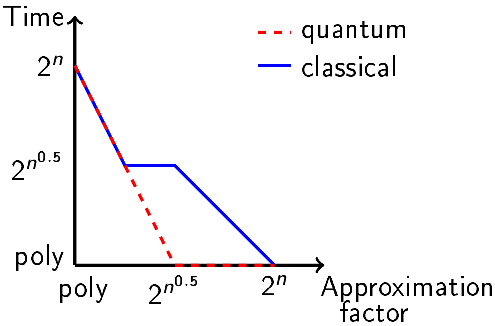
- Find $s \in L$ such that $\|s - t\| = \widetilde{O}(n^{\alpha})$
- Time:
  - $2^{\widetilde{O}(n^{1-2\alpha})}$ (query)
  - $+\ 2^{O(n)}$ (pre-processing)

[DLW19]: E. Doulgerakis, T. Laarhoven, and B. de Weger. Finding closest lattice vectors using approximate Voronoi cells. PQCRYPTO 2019.

[Ste19]: N. Stephens-Davidowitz. A time-distance trade-off for GDD with preprocessing – instantiating the DLW heuristic. arXiv 2019.

Conclusion

Approximation	Query time	Pre-processing
$2^{\widetilde{O}(n^{\alpha})}$	$2^{\widetilde{O}(n^{1-2\alpha})} + (\mathrm{poly}(n) \text{ or } 2^{\widetilde{O}(\sqrt{n})})$	$2^{O(n)}$



$+2^{O(n)}$ Pre-processing / Non-uniform algorithm

Extensions

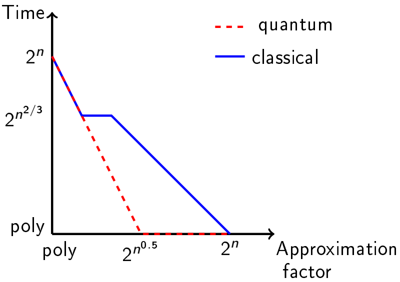
We can extend the algorithm to

- Non-principal ideals

Extensions

We can extend the algorithm to

- Non-principal ideals
- All number fields



Work in progress: "Euclidean division" over R

joint work with
Changmin Lee, Damien Stehlé and Alexandre Wallet

Finding short vectors in module lattices

(Principal) Ideals

Input: $a \in R$
Output: $x \in R$
such that $\|ax\|$ is small

(Free) Modules

Input: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$

Output: $(x, y) \in R^2$ such that

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \right\|$$

is small

If $R = \mathbb{Z}$: the LLL algorithm (or Gauss/Lagrange in dim 2)

LLL algorithm over $\mathbb{Z}$:

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{QR} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$
2. Reduce $r_{12} \leftarrow r_{12} \bmod r_{11}$
   ($\Rightarrow |r_{12}| \leq |r_{11}|/2$)
3. If $|r_{22}| \leq |r_{11}|/2$
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Adaptation to R

We need:

- A scalar product $\langle \cdot, \cdot \rangle$
  - and a norm $|\cdot|$
- A division (Step 2)
  - $|r_{12}| < (1 - \varepsilon)|r_{11}|$
  - swap condition $|r_{22}| < \varepsilon |r_{11}|$

Euclidean division

Over $\mathbb{Z}$

Input: $a, b \in \mathbb{Z}$
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Relax the requirement

Find $x, y \in R$ such that

- $\|xa + yb\| \leq \|a\|/2$
- $\|y\| \leq \mathrm{poly}(n)$

Using the Log space

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New objective

Find $x, y \in R$ such that

- $\|\operatorname{Log}(xa) - \operatorname{Log}(yb)\| \leq \varepsilon$
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Idea

Objective: find $x, y \in R$ s.t.

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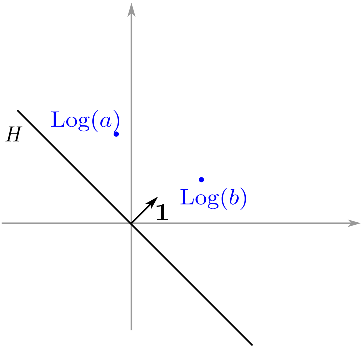

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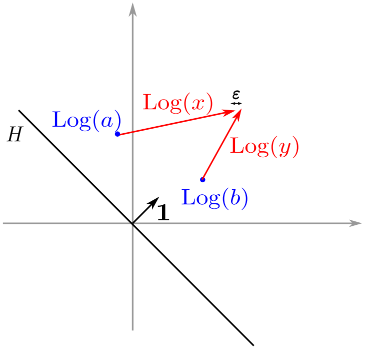

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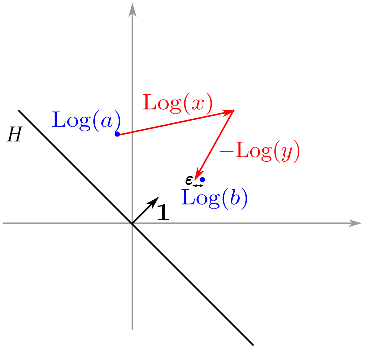

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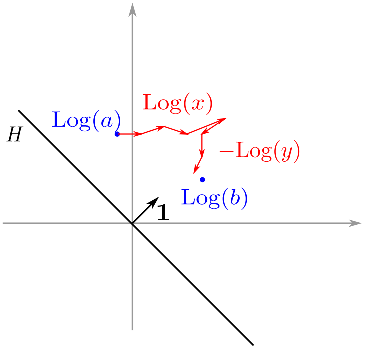

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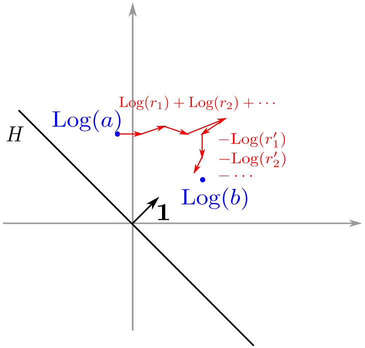

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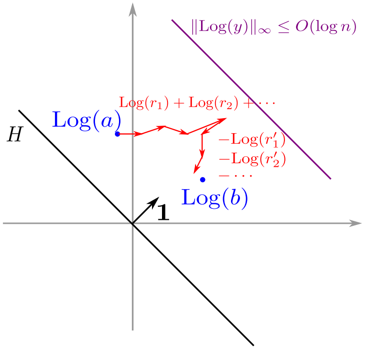

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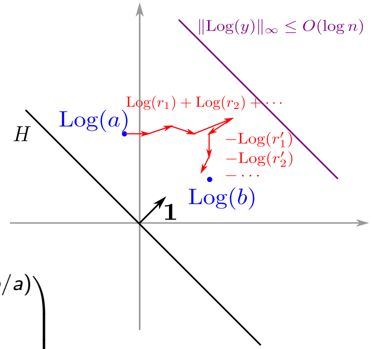

$$L = \begin{pmatrix} \mathrm{Log}\, r_1 & \mathrm{Log}\, r_2 & \cdots & \mathrm{Log}\, r_{n^2} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad t = \begin{pmatrix} \mathrm{Log}(b/a) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Conclusion

Difficulty

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Questions?