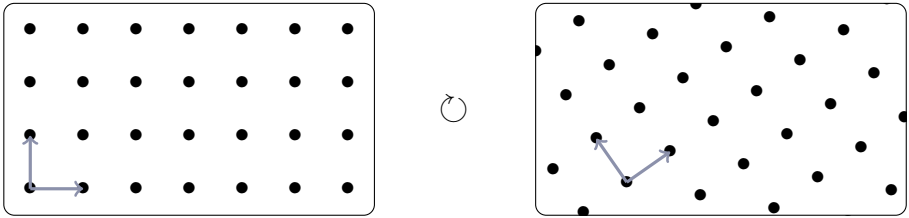
The (Module) Lattice Isomorphism Problem

Alice Pellet-Mary

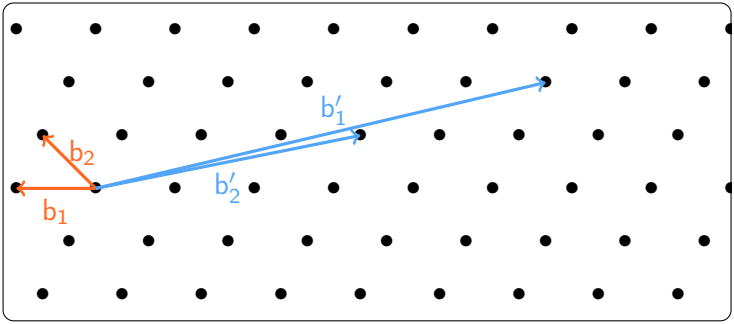
Oberwolfach workshop

January 2025

The lattice isomorphism problem



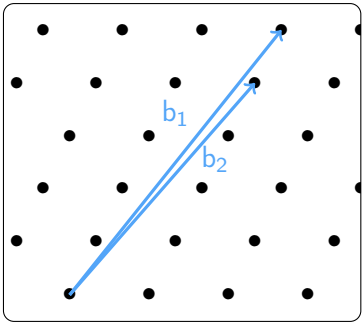
Lattices



- $\mathcal{L} = \{\sum_{i=1}^{n} x_i \mathbf{b}_i \mid \forall i,\ x_i \in \mathbb{Z}\}$ is a lattice
- $(\mathbf{b}_1, \ldots, \mathbf{b}_n) =: B \in \mathrm{GL}_n(\mathbb{R})$ is a basis (not unique)
- n is the dimension (or rank)

Short basis problem

Input:



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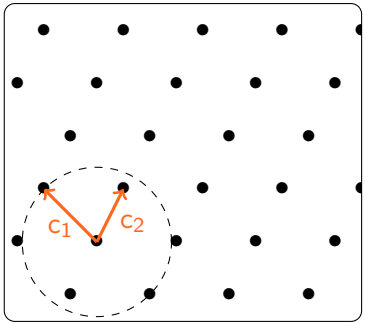
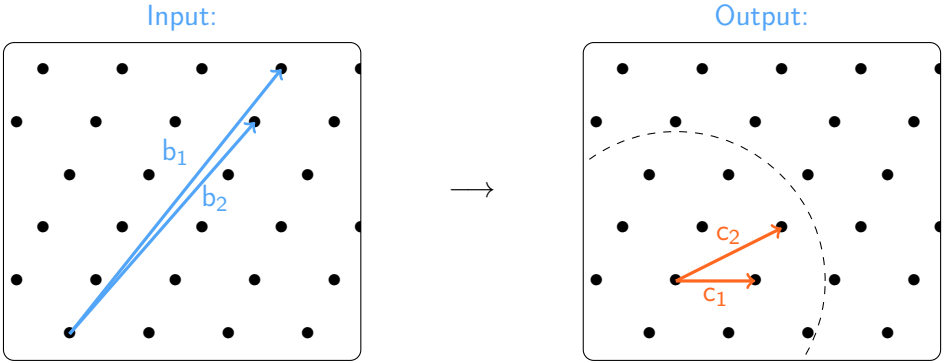


Shortest basis problem

$$\max_i \|c_i\| \leq \min_{B' \text{ basis of } \mathcal{L}} \left(\max_i \|b'_i\| \right)$$

Short basis problem



Approximate short basis problem

$$\max_i \|c_i\| \leq \gamma \cdot \min_{B' \text{ basis of } \mathcal{L}} \left(\max_i \|b'_i\| \right)$$

Hardness of the short basis problem

The short basis problem is hard

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The best known algorithms solving the short basis problem have complexity $\approx \exp(n)$ (for small approximation factors)

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  Analogy with factoring: factoring is hard
  - for some integers $\rightsquigarrow N = pq$ with p, q large primes is hard
  - but not for all integers $\rightsquigarrow N = 2p$ with p prime is easy

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How do we generate hard lattices?

The goal of all lattice assumptions

What we want: An algorithm `KeyGen` such that

- `KeyGen` computes
  - a random lattice $\mathcal{L}$
  - a short basis B_s of $\mathcal{L}$ (sk)
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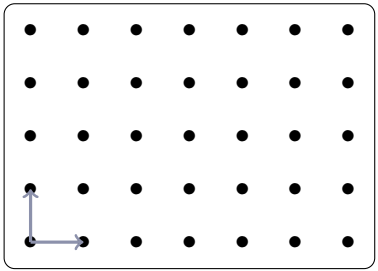
This is the purpose of all usual lattice assumptions:
LWE, SIS, NTRU, the lattice isomorphism problem (LIP), ...

The lattice isomorphism problem [DW22,BGPS23]



$\mathcal{L}_0 = \mathbb{Z}^n$

[DW22] Ducas and van Woerden. On the lattice isomorphism problem, quadratic forms, remarkable lattices and cryptography. Eurocrypt

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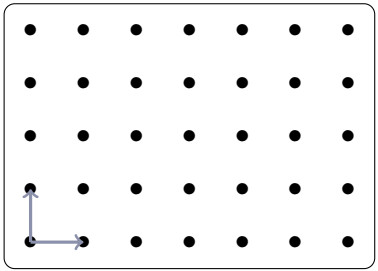

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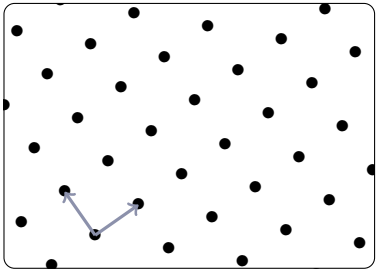
rotate

(choose O orthogonal matrix)



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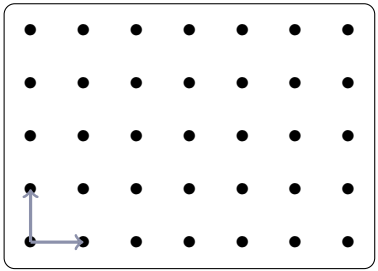

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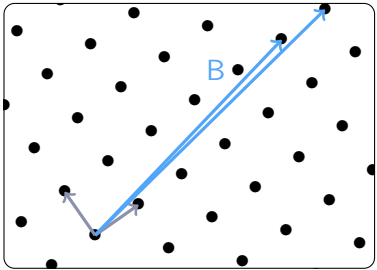

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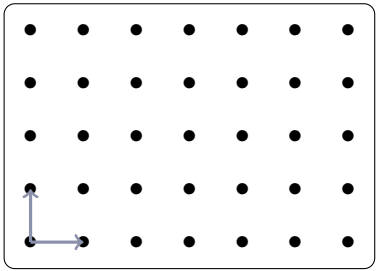

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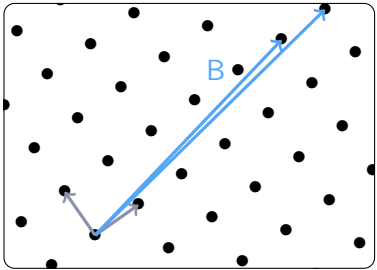

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Lattice Isomorphism Problem (LIP) assumption

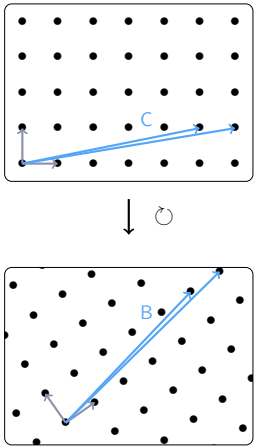
recovering O from B is hard

$\Leftrightarrow$ computing a shortest basis of $\mathcal{L}$ is hard

Equivalent formulation with Gram matrices

Lattice isomorphism problem: Given $B = O \cdot C$ with
- $O \in O_n(\mathbb{R})$ orthogonal
- C a basis of $\mathbb{Z}^n$

Find O (equivalently: find C)



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$G = B^T B = C^T (O^T O) C = C^T C$

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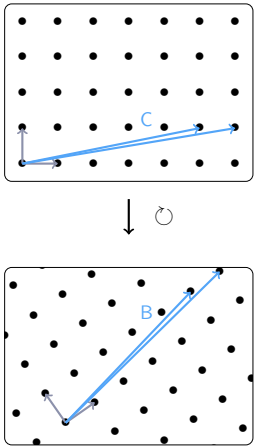

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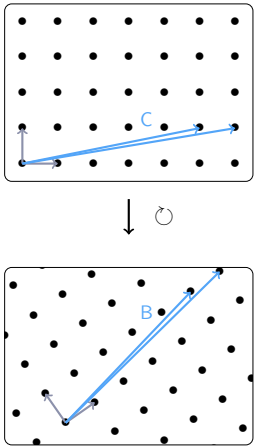

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Example:

- $C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$
- $O = \begin{pmatrix} 0.5 & 0.87 \\ 0.87 & -0.5 \end{pmatrix}$
- $B = \begin{pmatrix} 3.96 & 4.83 \\ -1.13 & -1.63 \end{pmatrix}$
- $G = \begin{pmatrix} 17 & 21 \\ 21 & 26 \end{pmatrix}$
  $= C^T C = B^T B$

Given G, recover $C \in \mathbb{Z}^{2\times 2}$ with $\det(C) = \pm 1$ such that $C^T C = G$

Section's conclusion

The lattice isomorphism problem (LIP):

- allows to generate random lattices
- where the short basis problem is believed to be hard to solve
- together with a short secret basis

We can use LIP to build crypto [DW22,BGPS23,DPPW23]
(see next section)

[DW22] Ducas and van Woerden. On the lattice isomorphism problem, quadratic forms, remarkable lattices and cryptography. Eurocrypt

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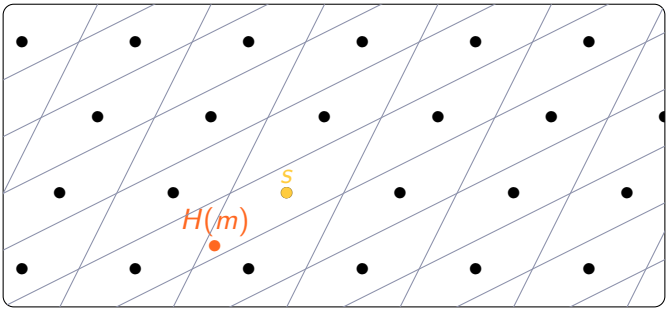
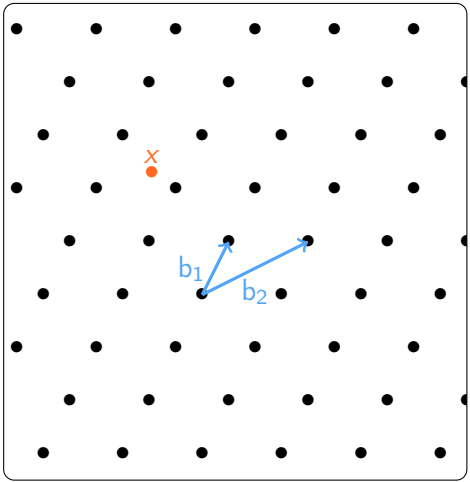
Hawk signature



Finding a close vector using a short basis



Input: $x = 3.7 \cdot \mathbf{b}_1 - 1.4 \cdot \mathbf{b}_2$

Objective: find $s \in \mathcal{L}$ close to x

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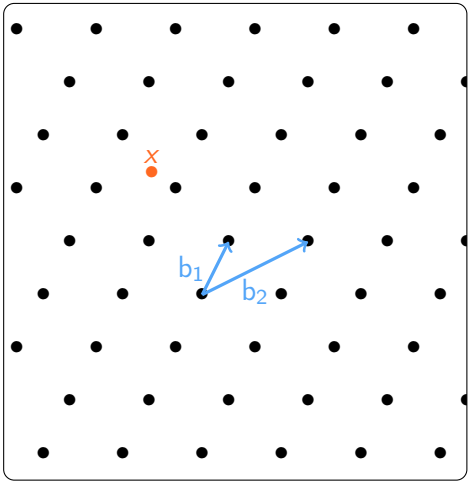

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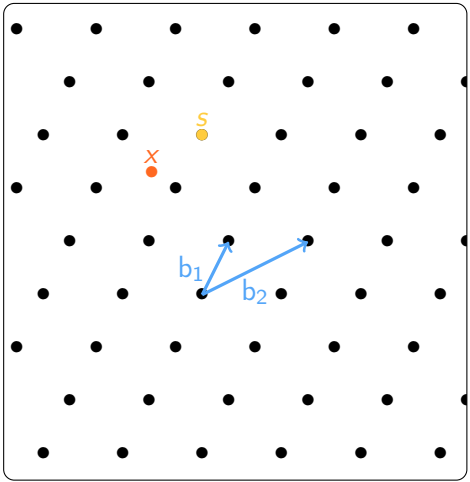

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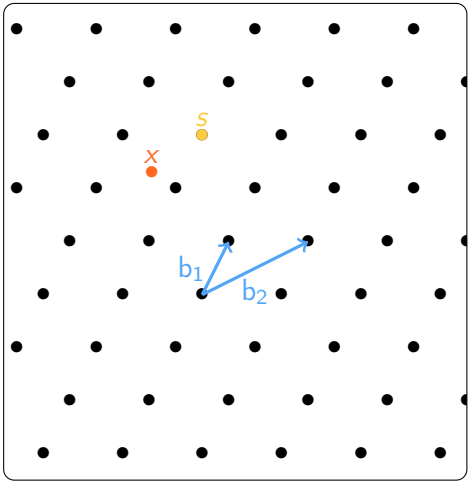

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The smaller the basis, the closer the solution

(called Babai's round-off algorithm)

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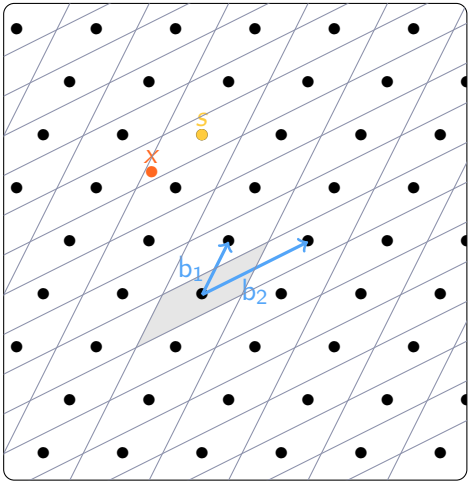

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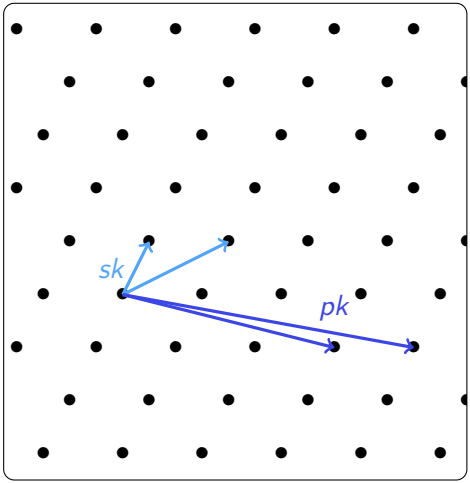
$$= \left\{ x_1\mathbf{b}_1 + x_2\mathbf{b}_2 \;\middle|\; |x_i| \leq \tfrac{1}{2} \right\}$$

Hash-and-sign framework [GGH97]



KeyGen:

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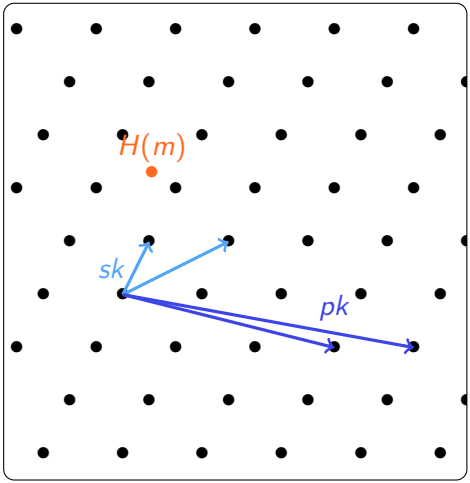
[GGH97] Goldreich, Goldwasser, and Halevi. Public-key cryptosystems from lattice reduction problems. Crypto

Hash-and-sign framework [GGH97]



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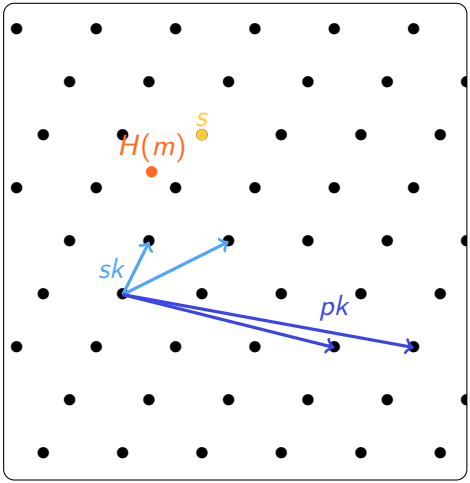

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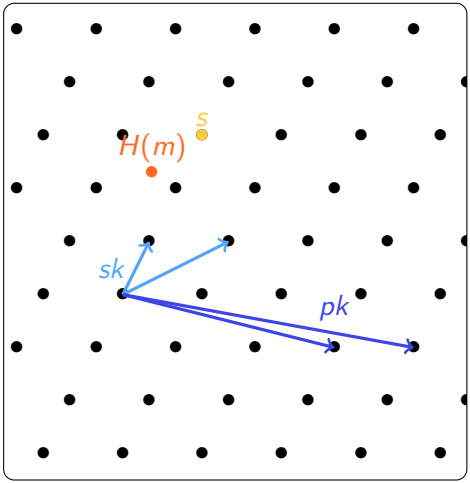

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Verify(m, s, pk):

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

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Hawk signature

Warning: The scheme from previous slide is insecure [NR06] but can be fixed [GPV08]

[NR06] Nguyen and Regev. Learning a parallelepiped: Cryptanalysis of GGH and NTRU signatures. J. Cryptology

[GPV08] Gentry, Peikert, and Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC.

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Hawk signature (NIST submission)

Hash-and-sign framework
+
hard lattice from lattice isomorphism problem

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Advantages of Hawk

Hash-and-sign framework can be instantiated with other assumptions

Assumption	LIP (lattice isomorphism problem)		
Construction's name	Hawk		

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[Ajt96] Ajtai. Generating hard instances of lattice problems. STOC.

[GPV08] Gentry, Peikert, Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC

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Hash-and-sign framework

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- simple to implement (no floating points → big advantage on Falcon)

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Other constructions from lattice isomorphism problem

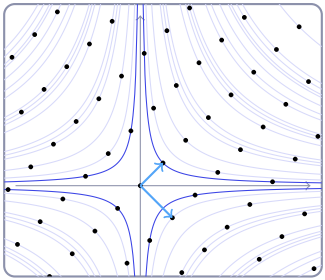
- key exchange [DW22,BGPS23]
- public key encryption [ARW24]

[DW22] Ducas and van Woerden. On the lattice isomorphism problem, quadratic forms, remarkable lattices and cryptography. Eurocrypt

[BGPS23] Bennett, Ganju, Peetathawatchai, Stephens-Davidowitz. Just how hard are rotations of $\mathbb{Z}^n$? [...] Eurocrypt

[ARW24] Ackermann, Roux-Langlois, Wallet. Public-key encryption from the lattice isomorphism problem. WCC presentation.

The Module Lattice Isomorphism Problem



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- allows faster operations

Number fields

Number field: $K = \mathbb{Q}[X]/P(X)$ (P irreducible, $\deg(P) = d$)

- $K = \mathbb{Q}$
- $K = \mathbb{Q}[X]/(X^d + 1)$ with $d = 2^\ell$ $\rightsquigarrow$ power-of-two cyclotomic field
- $K = \mathbb{Q}[X]/(X^d - X - 1)$ with d prime $\rightsquigarrow$ NTRUPrime field

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Ring of integers: $\mathcal{O}_K \subset K$, for this talk $\mathcal{O}_K = \mathbb{Z}[X]/P(X)$

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The canonical embedding

$(K = \mathbb{Q}[X]/P(X), \quad \alpha_1, \cdots, \alpha_d$ complex roots of $P(X))$

Canonical embedding: $\sigma : \begin{array}{ccc} K & \to & \mathbb{C}^d \\ a(X) & \mapsto & (a(\alpha_1), \cdots, a(\alpha_d)) \end{array}$

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- we can see K as a subset of $\mathbb{C}^d$
- this induces a geometry on K (using the hermitian norm in $\mathbb{C}^d$): for $a \in K$

$$\|a\| := \|\sigma(a)\|_2 = \sqrt{\overline{\sigma(a)}^T \sigma(a)}.$$

(Free) module:

$$M = \{B \cdot x \mid x \in {\mathcal{O}_K}^k\} \text{ for some matrix } B \in {\mathcal{O}_K}^{k \times k} \text{ with } \det_K(B) \neq 0$$

- k is the module rank
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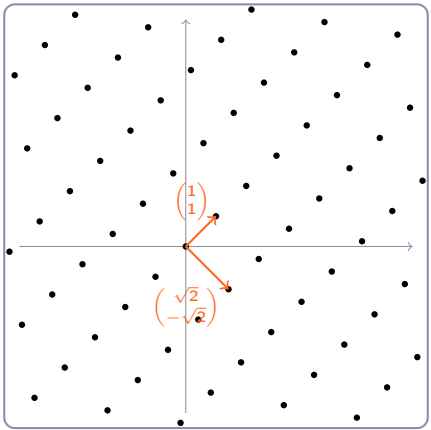
$\sigma(M)$ is a lattice: of $\mathbb{Z}$-rank $n := d \cdot k$, included in $\mathbb{C}^n$

$\Rightarrow \sigma(M)$ is called a module lattice

An example

$K = \mathbb{Q}[X]/(X^2+2)$, $\mathcal{O}_K = \mathbb{Z}[X]/(X^2+2)$, $\sigma : a + bX \mapsto \begin{pmatrix} a+b\sqrt{2} \\ a-b\sqrt{2} \end{pmatrix}$, $\mathcal{L} = \sigma(\mathcal{O}_K)$



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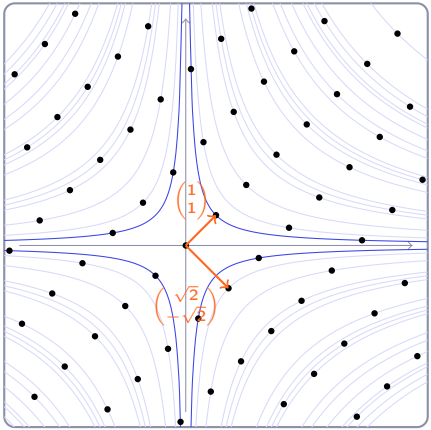

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Hawk relies on
module-LIP for the module ${\mathcal{O}_K}^2$, in a power-of-two cyclotomic field
($K = \mathbb{Q}[X]/(X^d + 1)$ with $d = 512$ or $d = 1024$)

Section's conclusion

Module lattices: module (algebraic object) + lattice (geometric object)

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- smaller storage size required, faster operations
  - more efficient cryptographic protocols
  - used in all standardized lattice-based schemes (Kyber, Dilithium, Falcon)

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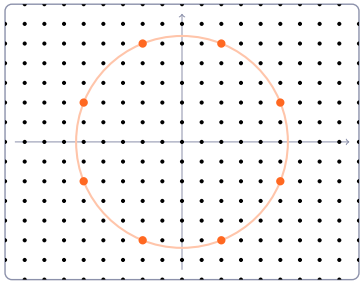
Drawbacks:

- maybe more efficient attacks using algebraic structure? (see next section)

Cryptanalysis of module-LIP



Cryptanalysis status

Lattice isomorphism problem (unstructured lattices)

- studied by mathematicians for ≥ 25 years (before crypto) [PS97]
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- for $k \geq 2$, best algorithm was thought to be the same as in the unstructured case (k = module rank)
- recent works show that for $k = 2$ and certain number fields, the module structure can be exploited! (this does not break Hawk so far) [MPPW24,LJPW24,CFMPW24,EP24]

[MPPW24] Mureau, Pellet-Mary, Pliatsok, Wallet. Cryptanalysis of rank-2 module-LIP in Totally Real Number Fields. Eurocrypt.

[LJPW24] Luo, Jiang, Pan, Wang. Cryptanalysis of Rank-2 Module-LIP with Symplectic Automorphisms. Asiacrypt.

[CFMPW24] Chevignard, Fouque, Mureau, Pellet-Mary, Wallet. A reduction from Hawk to the principal ideal problem in a quaternion algebra. Eprint.

[EP24] Espitau, Pliatsok. On hermitian decomposition lattices and the module-LIP problem in rank 2. Eprint.

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Objective: Given $\mathcal{L}$ and $\mathcal{L}' = O \cdot \mathcal{L}$ (O orthogonal), recover O

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Cost:

- at least as high as computing shortest vectors in $\mathcal{L}$ and $\mathcal{L}'$
- in very special worst cases, the matching part 2. may dominate
  (the number of shortest vectors may be huge)

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Cryptanalysis of module-LIP: when P has a real root

Notations: $K = \mathbb{Q}[X]/P(X)$, $\mathcal{O}_K = \mathbb{Z}[X]/P(X)$

Objective: Given $G := \overline{C}^T C$, recover C (where $C = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in {\mathcal{O}_K}^{2\times 2}$)

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Key point: if P has at least 1 real root, then from G we can recover

$$C^T C = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

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New objective: given α $(= a^2 + b^2)$, find all solutions $(x, y) \in {\mathcal{O}_K}^2$ of the equation

$$x^2 + y^2 = \alpha$$

Solving sum of two square equations

For simplicity, take $\mathcal{O}_K = \mathbb{Z}$ (similar strategy for general $\mathcal{O}_K$)

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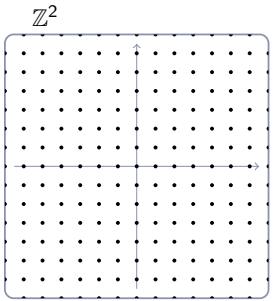

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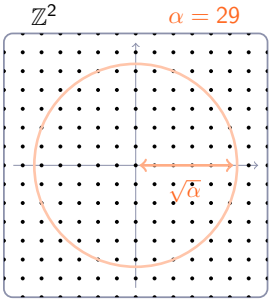

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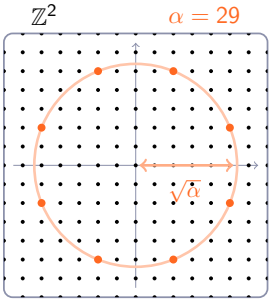

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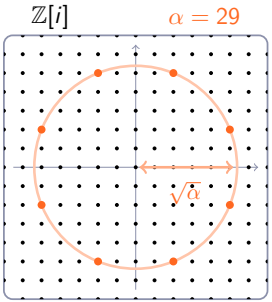
Reformulate:

- take $\mathbb{Z}[i] \subseteq \mathbb{C}$
- for $z = x + iy \in \mathbb{Z}[i]$, $\bar{z}z = x^2 + y^2$
- solve $\bar{z}z = \alpha$, for $z \in \mathbb{Z}[i]$



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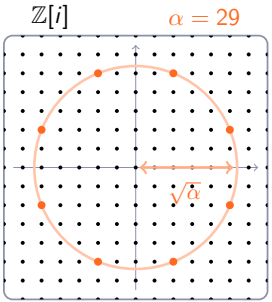

Conclude: solve $\bar{z}z = \alpha$ using Gentry-Szydlo algorithm [GS02,LS19]

[GS02] Gentry, Szydlo. Cryptanalysis of the revised NTRU signature scheme. Eurocrypt

[LS19] Lenstra, Silverberg. Testing isomorphism of lattices over CM-orders. Journal on Computing

Fields with one real embedding: summary

Notations: $K = \mathbb{Q}[X]/P(X)$, $\mathcal{O}_K = \mathbb{Z}[X]/P(X)$, k module rank

Assumptions:

- P has (at least) one real root
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Heuristic quantum poly time algorithm for module-LIP [MPPW24,APW25]

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$\Rightarrow$ take $\mathcal{H} = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + ij\mathbb{Q}$ (quaternion algebra, $i^2 = j^2 = -1$, $ij = -ji$)

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Cyclotomic fields (continued)

Module-LIP $\leq$ solving $z\bar{z} = \alpha$ in $\mathcal{H} = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + ij\mathbb{Q}$

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Do we have a Gentry-Szydlo algorithm in $\mathcal{H}$?

- not for the moment (when the degree of P is large)
- cannot conclude the attack in the cyclotomic case

Summing up

Hardness of module-LIP in various fields (modules of rank 2, $K = \mathbb{Q}[X]/P$)

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Summing up

Hardness of module-LIP in various fields (modules of rank 2, $K = \mathbb{Q}[X]/P$)

- If P has (at least) one real root: heuristic quantum poly time attack
- NTRUPrime fields: heuristic classical poly time attack
- Cyclotomic fields: no attack to far...
  ...but a reduction to Gentry-Szydlo in a quaternion algebra

Conclusion

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  - possible strategy: rephrase LWE-based constructions in a more geometric way

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Thank you