Euclidean lattices in cryptography

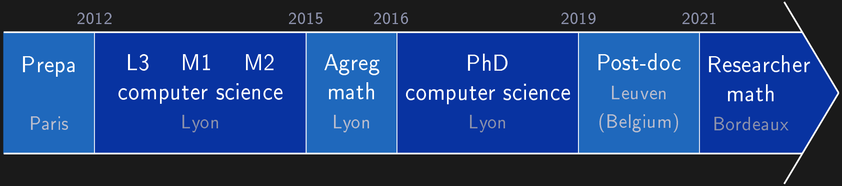
Alice Pellet--Mary

CNRS and Université de Bordeaux

MARGAUx PhD days

Bordeaux

What I did until now



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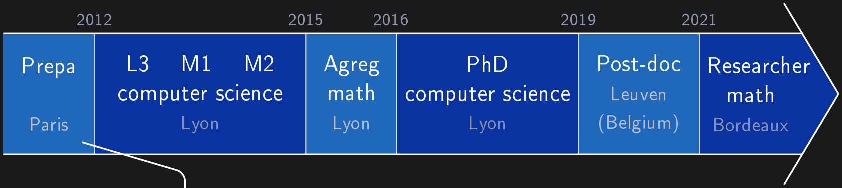

"If you want to do research, go for computer science, they have more money"

math teacher

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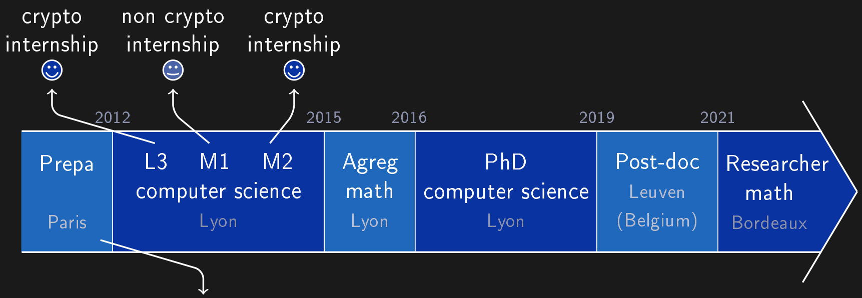

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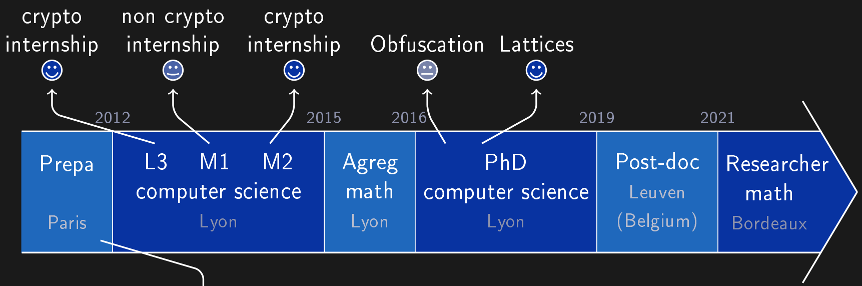

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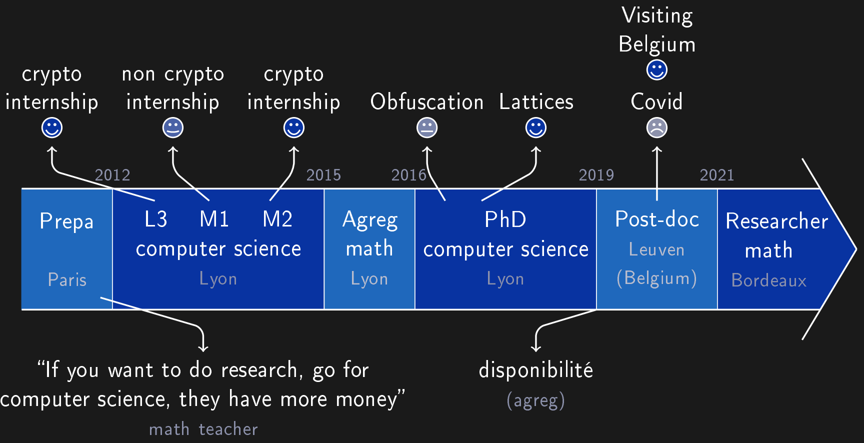

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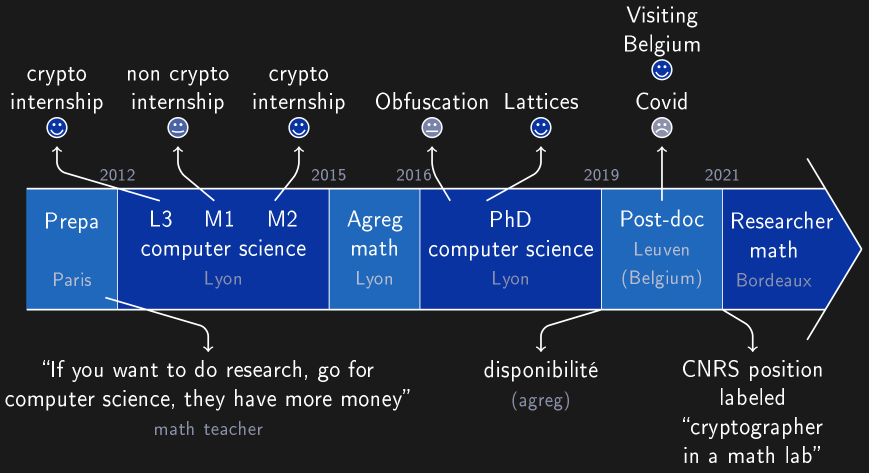
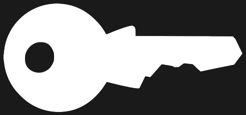


Cryptography



Cryptography

(Cryptology =) Cryptography = science of secrets

Examples: encryption, signatures, homomorphic encryption, e-voting . . .

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A cryptographic primitive is mathematically defined by

- some correctness properties
- some security properties
  - need to model the attacker

Encryption: symmetric

Alice	Bob
s $(\in \mathbb{Z})$	s $(\in \mathbb{Z})$

Encryption: symmetric

Alice

s $(\in \mathbb{Z})$

$c = \mathrm{Enc}(m, s)$

(message $m \in \{0, 1\}$)

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Security: (against chosen plaintext attacks)
for any polynomial time algorithm $\mathcal{A}$,

$$\Pr_{m \leftarrow \mathcal{U}(\{0,1\})}\Big(\mathcal{A}(\mathrm{Enc}(m,s)) = m\Big) = 1/2 + \varepsilon$$

(usually require $|\varepsilon| \leq 2^{-128}$)

Encryption: asymmetric [DH76, RSA78]

Alice		Bob
	$\xleftarrow{\;pk\;}$	sk, pk $(\in \mathbb{Z})$

[DH76] Diffie, Hellman. New Directions in Cryptography.

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Examples: factoring, discrete logarithm, ...

Definition: an algorithmic problem is hard if there is no algorithm solving it in polynomial time

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asymmetric encryption | signature | homomorphic encryption | ...

error correcting codes | lattices | isogenies

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(Supposedly hard) algorithmic problems

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(Supposedly hard) algorithmic problems in a quantum world

My research: classify lattice problems

Lattice problem 2

Lattice problem 1

Lattice problem 3

Lattice problem 4

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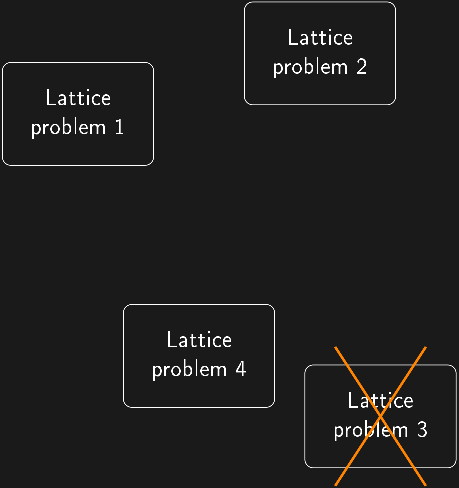

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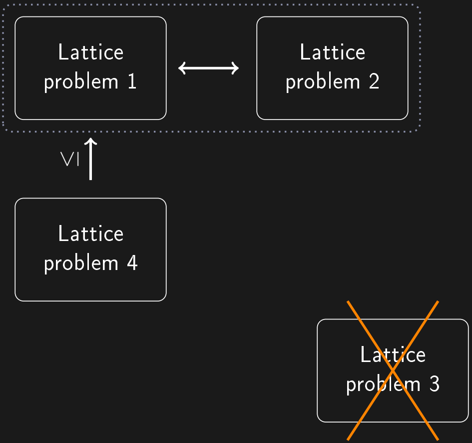

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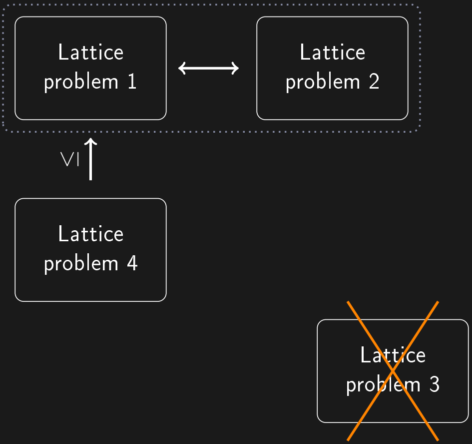
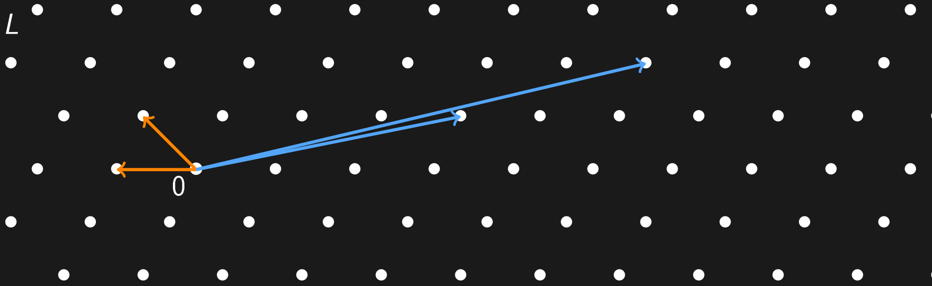


Tools:

- algorithms
- number theory
- a little programming
  (just toy examples)

Lattices and algorithmic problems

Lattices

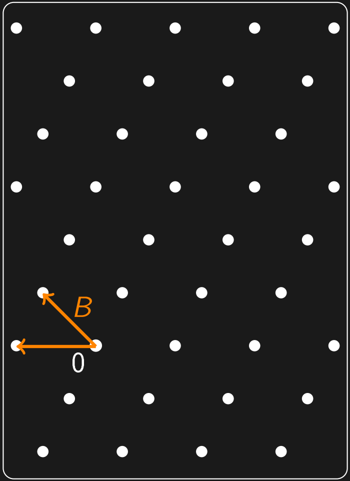


- $L = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
- $B \in \mathrm{GL}_n(\mathbb{R})$ is a basis
- n is the dimension of L

Representing a lattice

Representation of a lattice L:

a basis $B \in \mathbb{Z}^{n \times n}$ of L



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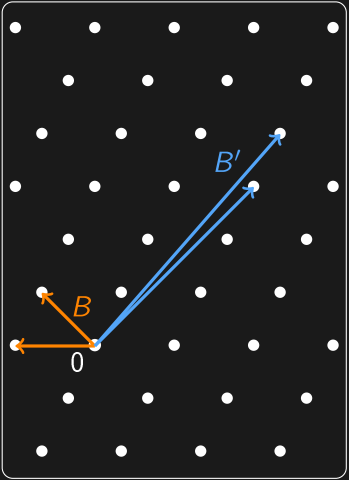

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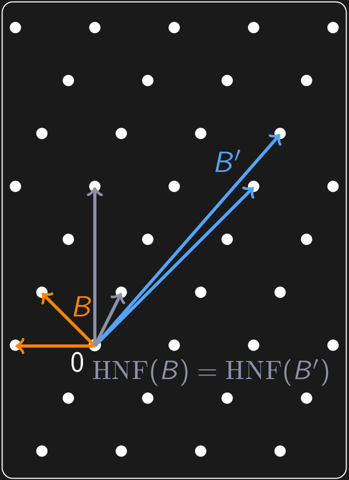
Solution: take the Hermite Normal Form (HNF) of any B

- it is unique $(\mathrm{HNF}(B) = \mathrm{HNF}(B'))$
- it is efficiently computable



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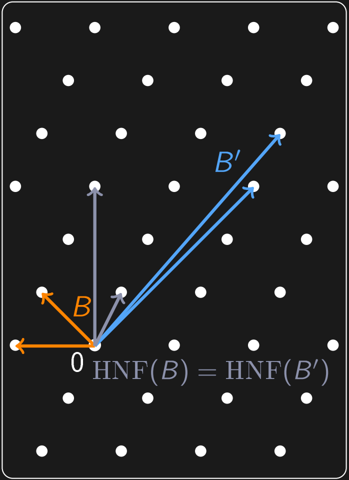
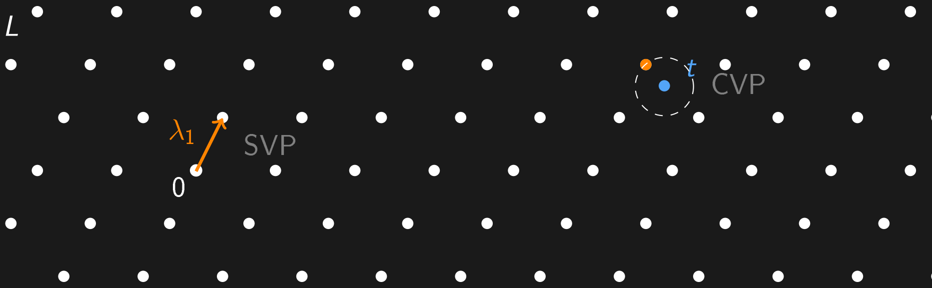
$\Rightarrow$ canonical representation of L
(i.e., worse basis ever)



Algorithmic problems



SVP : Shortest Vector Problem
(input: HNF basis of L)

CVP : Closest Vector Problem
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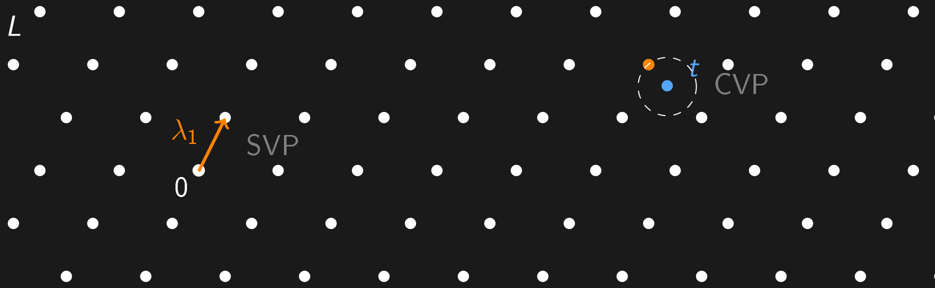

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Supposedly hard to solve when n is large
(even with a quantum computer)

How hard is SVP/CVP?

In theory: best algorithm has asymptotic complexity $2^{c \cdot n + o(n)}$
(for some $c \approx 0.292$, or $c \approx 0.265$ for quantum computers [Laa15])

$\Rightarrow$ not polynomial

[Laa15] Laarhoven. Search problems in cryptography.

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- from $n = 500$ to $n = 1000 \rightsquigarrow$ cryptography

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The zoo of lattice problems

Exact solution vs Approximation

- find a shortest vector
- find a vector $\leq \gamma$

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Exact solution	vs	Approximation
▶ find a shortest vector		▶ find a vector $\leq \gamma$
Search	vs	Decision
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Search vs Decision

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Worst-case vs Average-case

- find a short vector in any possible input lattice L
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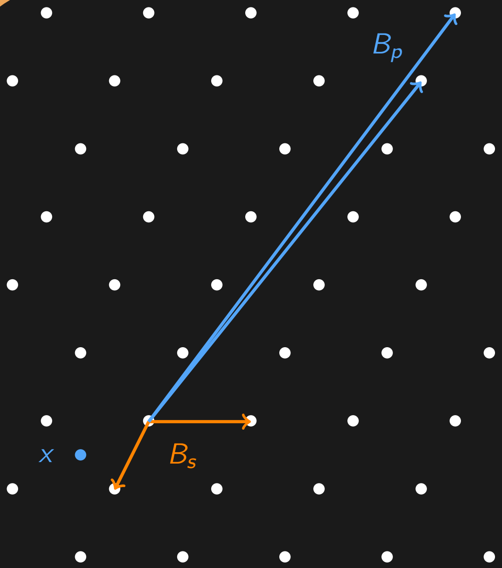
Plain lattices	vs	**Algebraic lattice**
▶ find a short vector in a lattice over $\mathbb{Z}$		▶ find a short vector in a lattice over $\mathcal{O}_K$ (ring of integers of number field K)

Digression: building cryptography from lattices

Asymmetric encryption from lattices



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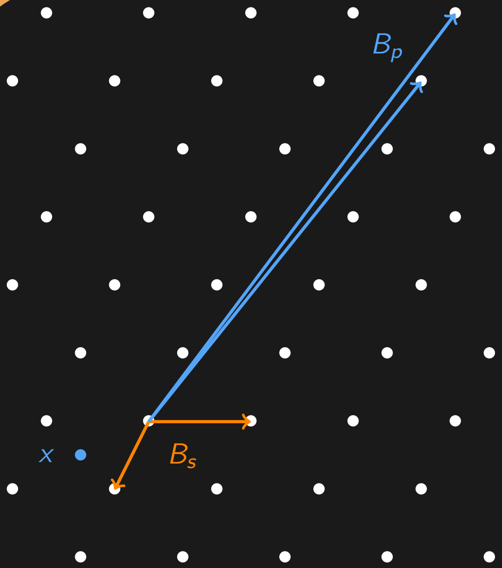

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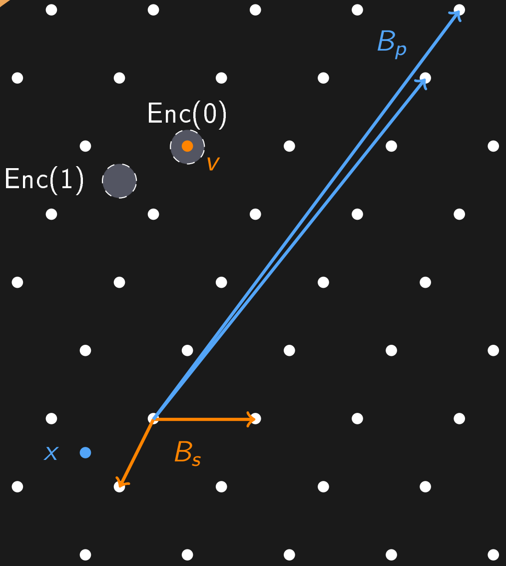

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- sample small $e \in \mathbb{R}^n$
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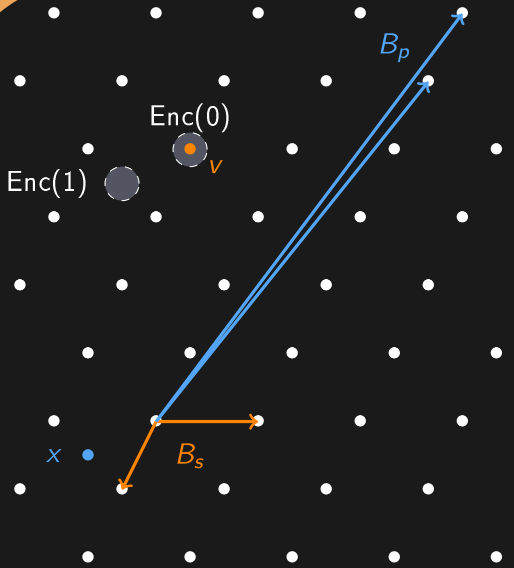

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Decryption(c, sk):

- find $w \in L$ closest to c
- if c is very close to w, return $m = 0$
- otherwise return $m = 1$

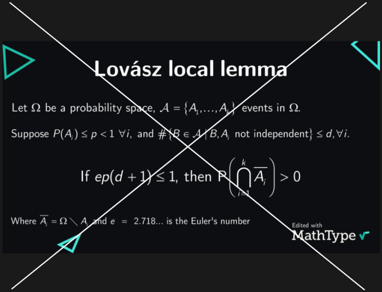
Correctness and security

Theorem

The encryption construction is correct and secure assuming that the problem decision-CVP is hard.

decision-CVP: given the HNF basis of L and a target t, decide whether t is close to a point of L or not.

The LLL algorithm



Objective

The LLL algorithm

- runs in polynomial time
- finds a vector $v \in L$ with $\|v\|_2 \leq 2^n \cdot \lambda_1(L)$ $\quad (\lambda_1(L) = \min_{\substack{w \in L \\ w \neq 0}} \|w\|_2)$

[LLL82] A. K. Lenstra, H. W. Lenstra, L. Lovász. Factoring polynomials with rational coefficients.

Dimension 2: Lagrange-Gauss algorithm

video

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Theorem: the algorithm

- finds a shortest vector of L
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Larger dimension: LLL algorithm

Input: basis $B = (b_1, \ldots, b_n)$

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- while there exist i such that $\|b_i\|_2 > \lambda_1(L_i)$
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- finds $v \in L$ with $\|v\|_2 \leq 2^n \cdot \lambda_1(L)$
- does not run in polynomial time

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Input: basis $B = (b_1, \dots, b_n)$

Main idea: improve the basis locally on blocks of dimension 2
(using Lagrange-Gauss algorithm)

Algorithm:

- while there exist i such that $\|b_i\|_2 > 4/3 \cdot \lambda_1(L_i)$
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  - run Lagrange-Gauss on L_i

This algorithm

- finds $v \in L$ with $\|v\|_2 \leq 2^n \cdot \lambda_1(L)$
- runs in polynomial time

LLL over a number field?

Can we adapt LLL to lattices over $\mathcal{O}_K$?

[LPSW19] Lee, Pellet-Mary, Stehlé, Wallet. An LLL algorithm for module lattices.

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- computing the pseudo-division is super costly ☹

[LPSW19] Lee, Pellet-Mary, Stehlé, Wallet. An LLL algorithm for module lattices.

LLL over a number field?

Can we adapt LLL to lattices over $\mathcal{O}_K$?

For LLL we need:

- QR factorization $\Rightarrow$ ok
- Euclidean division $\Rightarrow$ not ok ($\mathcal{O}_K$ is usually not Euclidean)

(Partial) solution: use pseudo-euclidean division
($|au + bv| < 1/2 \cdot |a|$ and $|v|$ not too big instead of $|au + v| < 1/2 \cdot |a|$)

- the LLL algorithm still works with pseudo-division ☺
- computing the pseudo-division is super costly ☹

$\Rightarrow$ we obtain LLL over $\mathcal{O}_K$ but not polynomial time anymore

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Conclusion

Take-away: crypto is fun!

(It is a good way to do nice math and have founding)

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Thank you