Cryptography, hard problems and algorithmic number theory

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CNRS et Université de Bordeaux

Présentation scientifique dans le cadre du DOR 2023

Public key cryptography

Cryptographic primitives

public key encryption	signature	homomorphic encryption	$\dots$

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public key encryption | signature | homomorphic encryption | $\cdots$

error correcting codes | lattices | isogenies

factoring | discrete logarithm | $\cdots$

(Supposedly intractable) algorithmic problems

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(Supposedly intractable) algorithmic problems

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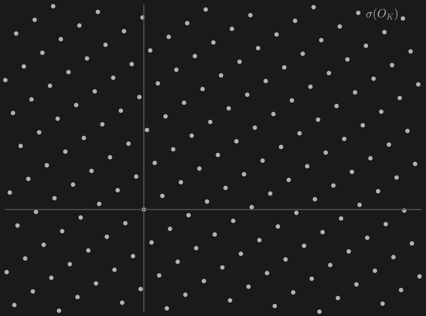

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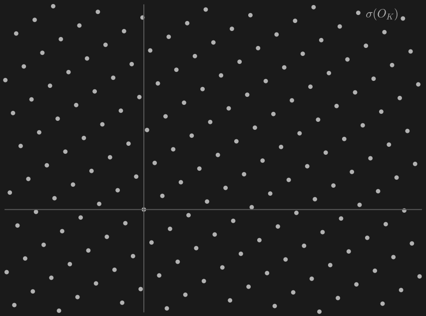

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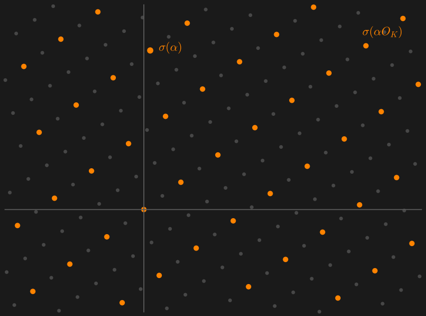

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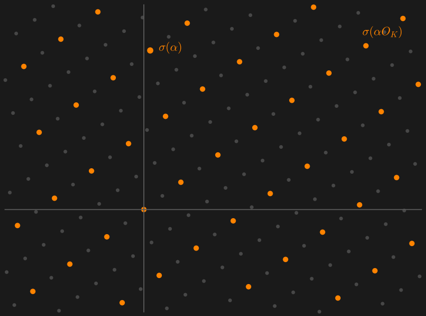

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ideal-Shortest Vector Problem (ideal-SVP): given α, find $\alpha r \in \alpha O_K$ such that $\|\sigma(\alpha r)\|_2$ is as small as possible (and $\neq 0$)

Short vectors in lattices

ideal-SVP is a special case of the

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- Known algorithms scale badly (exponentially) with the dimension n
  - $\rightsquigarrow$ $n = 2$ ☺ (by hand)
  - $\rightsquigarrow$ $n \approx 60$ ☺ (personal laptop)
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Remark: there are families of nice lattices in which SVP is easy

Exploiting the algebraic structure

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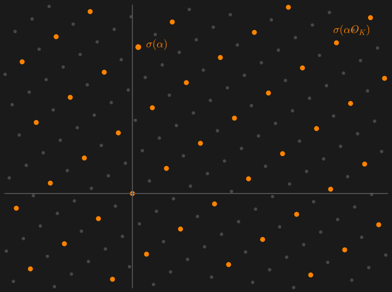
Objective: find a short element in $\alpha O_K = \{\alpha r \mid r \in O_K\}$



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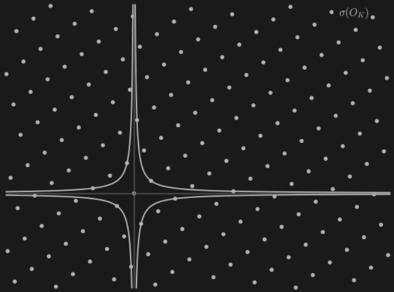

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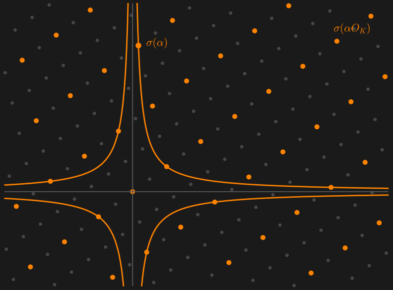

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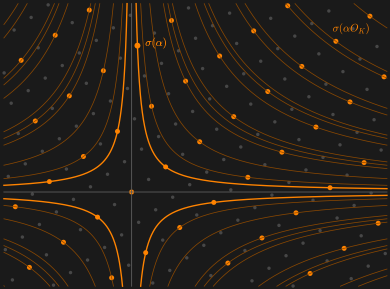

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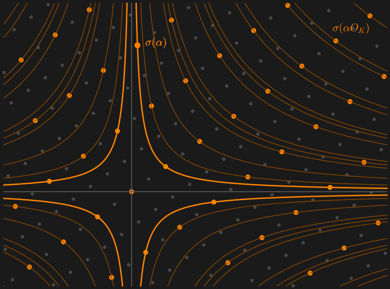

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Idea:

- focus on finding $u \in O_K^\times$ s.t. $\|\sigma(\alpha u)\|$ is minimal
- take the $\log(|\cdot|)$ coordinate-wise (Log)
  - $\rightsquigarrow$ $\mathrm{Log}(\{\alpha u \mid u \in O_K^\times\}) = \mathrm{Log}(\alpha) + \mathrm{Log}(O_K^\times)$ is a shifted lattice
  - $\rightsquigarrow$ for some K, $\mathrm{Log}(O_K^\times)$ is a nice lattice (nicer than $\sigma(\alpha O_K)$)
  - $\rightsquigarrow$ use lattice algorithms here!

Summing up

Motivation: cryptography

Tools:

- algorithmic number theory
- complexity theory

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Thank you