Approx-SVP in Ideal lattices with Pre-Processing

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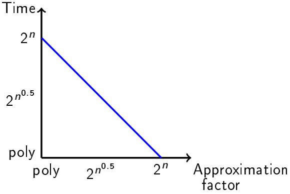
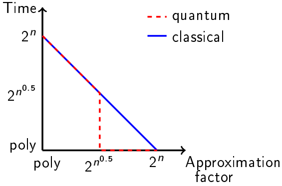
What is this talk about

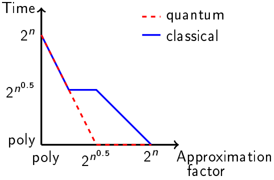
Time/Approximation factor trade-off for SVP in ideal lattices:



BKZ algorithm

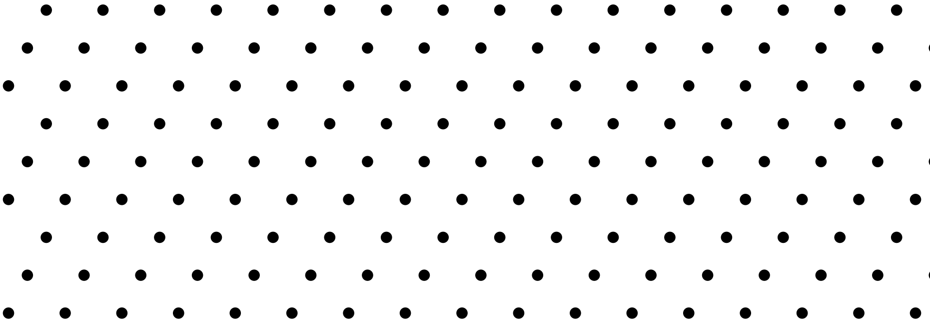


[CDPR16,CDW17]



This work
(with $2^{O(n)}$ pre-processing)

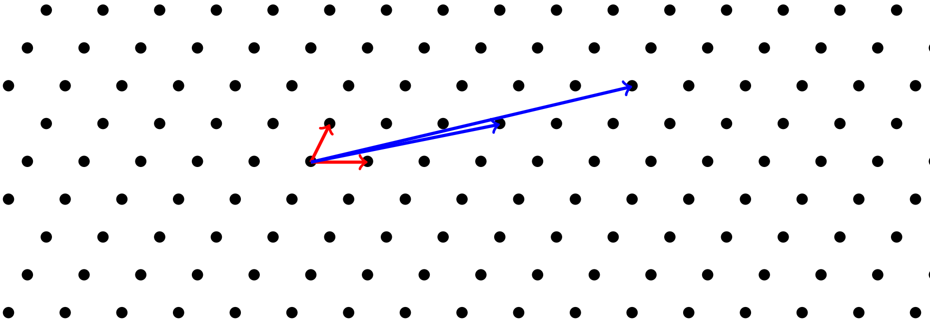
Lattices



Lattice

A lattice L is a discrete 'vector space' over $\mathbb{Z}$.

Lattices

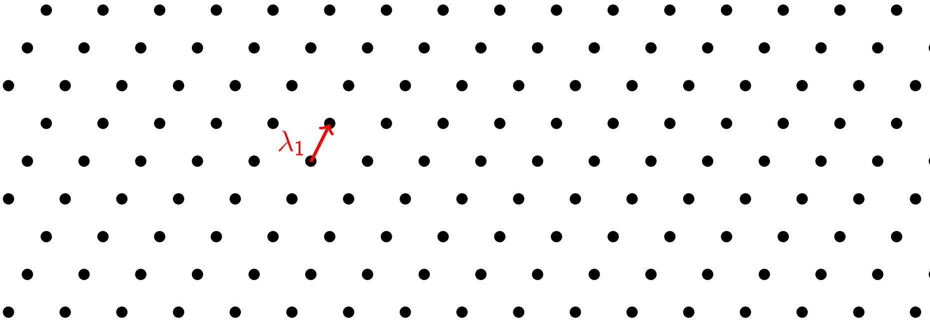


Lattice

A lattice L is a discrete 'vector space' over $\mathbb{Z}$.
A basis of L is an invertible matrix B such that $L = \{Bx \mid x \in \mathbb{Z}^n\}$.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 11 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

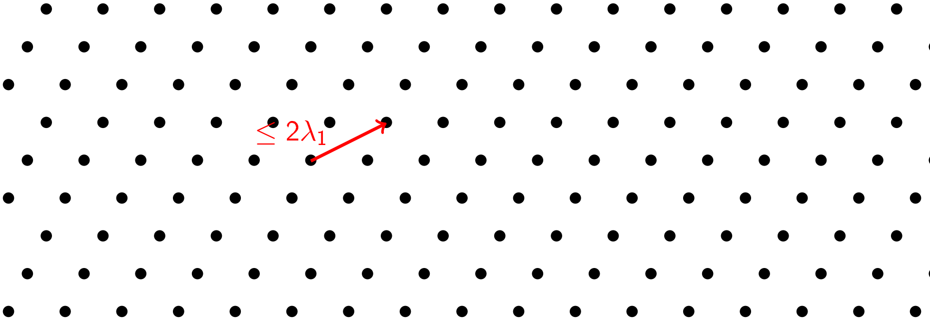
Lattices



Shortest Vector Problem (SVP)

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted λ_1.

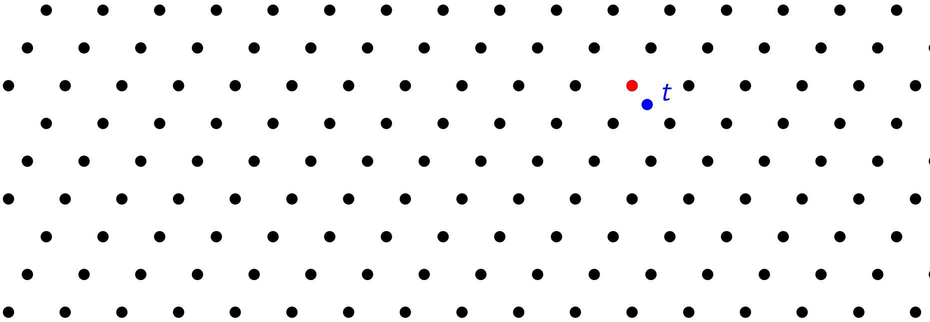
Lattices



Approximate Shortest Vector Problem (approx-SVP)

Find a short (in Euclidean norm) non-zero vector.
(e.g. of norm $\leq 2\lambda_1$).

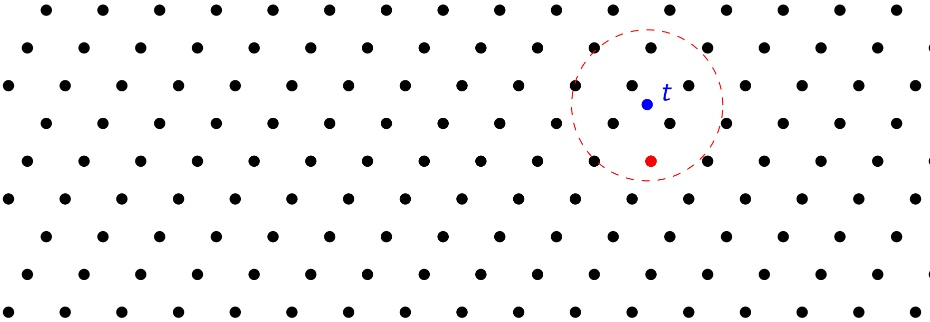
Lattices



Closest Vector Problem (CVP)

Given a target point t, find a point of the lattice closest to t.

Lattices



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Complexity of SVP/CVP

Applications

SVP and CVP in general lattices are conjectured to be hard to solve both quantumly and classically $\Rightarrow$ used in cryptography

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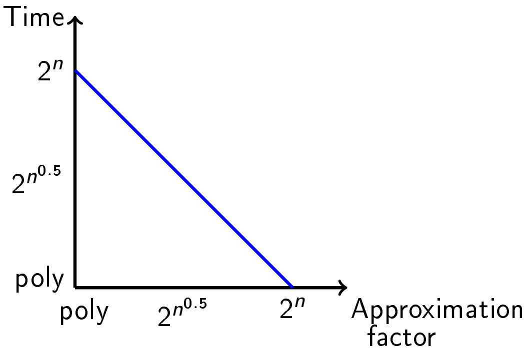
Best Time/Approximation trade-off for general lattices: BKZ algorithm



Structured lattices

Improve efficiency of lattice-based crypto using structured lattices.
$\Rightarrow$ E.g. ideal lattices

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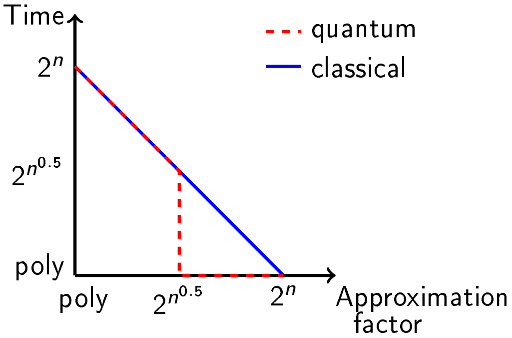
Is approx-SVP still hard when restricted to ideal lattices?

SVP in ideal lattices

[CDPR16,CDW17]: Better than BKZ in the quantum setting

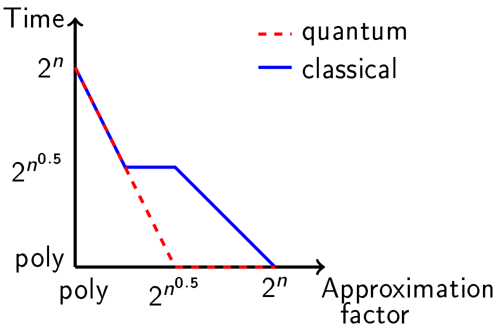


- Heuristic

[CDPR16] R. Cramer, L. Ducas, C. Peikert and O. Regev. Recovering Short Generators of Principal Ideals in Cyclotomic Rings, Eurocrypt.

[CDW17] R. Cramer, L. Ducas, B. Wesolowski. Short Stickelberger Class Relations and Application to Ideal-SVP, Eurocrypt.

This work



- Heuristic
- Pre-processing $2^{O(n)}$, independent of the choice of the ideal (non-uniform algorithm).

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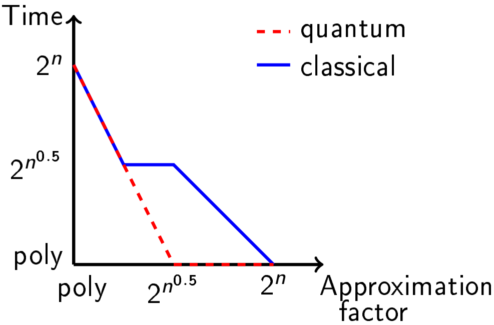

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Disclaimer: In this talk, only *principal* ideal lattices

Outline of the talk

First definitions

Notation

$R = \mathbb{Z}[X]/(X^n + 1)$ for $n = 2^k$

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Minkowski's embedding

- $\zeta \in \mathbb{C}$ primitive $2n$-th root of unity ($\zeta^{2n} = 1$)
- For $r \in R$, define

$$\sigma(r) = (r(\zeta), r(\zeta^3), \cdots, r(\zeta^{n-1})) \in \mathbb{C}^{n/2} \cong \mathbb{R}^n$$

Geometric and algebraic structures

Notation

$$\sigma(r) = (\widetilde{r_1}, \cdots, \widetilde{r_{n/2}}) \in \mathbb{C}^{n/2}$$

Geometric structure:

- Euclidean norm: $\|r\| = \sqrt{\sum_{i=1}^{n/2} |\widetilde{r_i}|^2}$
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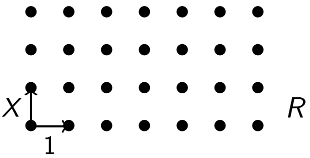

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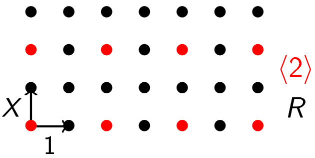

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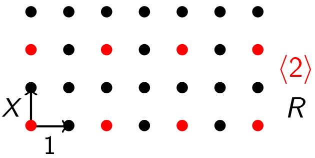

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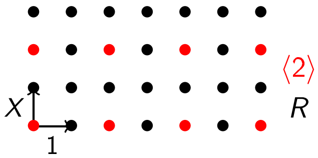

Algebraic structure:

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  - $\mathcal{N}(ab) = \mathcal{N}(a) \cdot \mathcal{N}(b)$ for all $a, b \in R$,
  - $\mathcal{N}(a) \geq 1$ and $\mathcal{N}(a) \in \mathbb{Z}$ for all $a \in R \setminus \{0\}$,
  - $\mathcal{N}(u) = 1 \iff u \in R^{\times}$.

Relations between algebraic/geometric structures

Reminder: $\sigma(r) = (\widetilde{r_1}, \cdots, \widetilde{r_{n/2}})$

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- Euclidean/algebraic norm:
  - $\|r\|$ small $\Rightarrow \mathcal{N}(r)$ relatively small.
  - $\mathcal{N}(r)$ small $\not\Rightarrow \|r\|$ relatively small (e.g. $(2^{-50}, 2^{50})$).

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- $\lambda_1(\langle g \rangle) = \mathrm{poly}(n) \cdot \mathcal{N}(g)^{1/n}$

Objective of this talk

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Given a basis of a principal ideal $\langle g \rangle$ and $\alpha \in (0, 1]$,
Find $r \in \langle g \rangle$ such that $\|r\| \leq 2^{\widetilde{O}(n^{\alpha})} \cdot \lambda_1 = 2^{\widetilde{O}(n^{\alpha})} \cdot \mathcal{N}(g)^{1/n}$.

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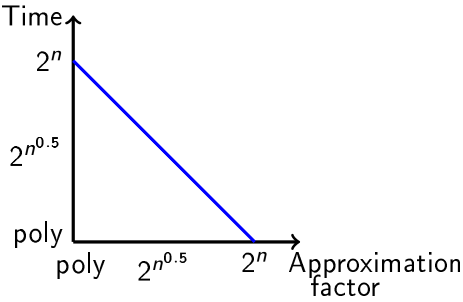
BKZ algorithm can do it in time $2^{\tilde{O}(n^{1-\alpha})}$, can we do better?



Outline of the talk

1. Definitions and objective
2. The CDPR algorithm
3. This work

Overview of the CDPR algorithm (on an idea of [CGS14])

Important points

- Large algebraic norm $\Rightarrow$ large Euclidean norm.
- In $\langle g \rangle$, the elements with the smallest algebraic norm are the generators.

[CGS14]: P. Campbell, M. Groves, and D. Shepherd. Soliloquy: A cautionary tale.

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The CDPR algorithm: find a generator with a smallest Euclidean norm

- Find a generator g_1 of $\langle g \rangle$
  - [BS16]: quantum time $\mathrm{poly}(n)$
  - [BEFGK17]: classical time $2^{\widetilde{O}(\sqrt{n})}$
- Find $u \in R^\times$ which minimizes $\|ug_1\|$.

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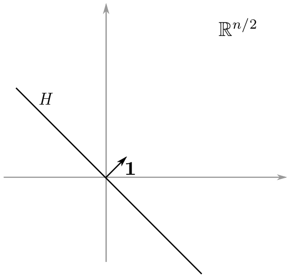
The Log unit lattice

Definitions

$$\begin{aligned} \mathrm{Log} : \sigma(R) &\to \mathbb{R}^{n/2} \\ (\widetilde{r_1}, \cdots, \widetilde{r_{n/2}}) &\mapsto (\log|\widetilde{r_1}|, \cdots, \log|\widetilde{r_{n/2}}|) \end{aligned}$$

Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^{\perp}$.



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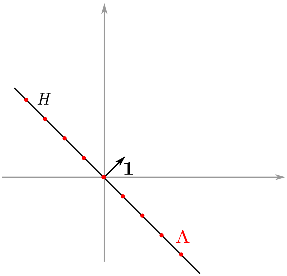
Theorem (Dirichlet)

$\Lambda := \mathrm{Log}(R^{\times})$ is a lattice included in H.



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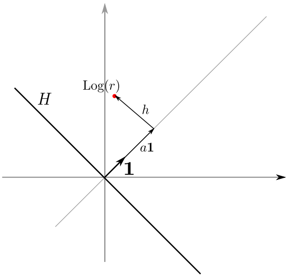

Write $\boxed{\mathrm{Log}(r) = h + a\mathbf{1}}$, with $h \in H$

- $\|r\| \leq \sqrt{n} \cdot 2^a \cdot 2^{\|h\|}$
- $a = \dfrac{\log|\mathcal{N}(r)|}{n}$

CDPR (upper bound)

Reminder ($\mathrm{Log}(r) = h + a\mathbf{1}$)

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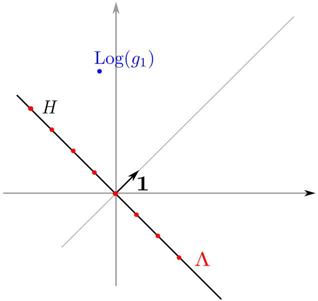

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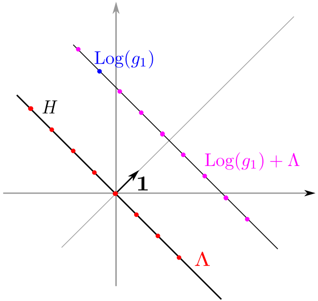

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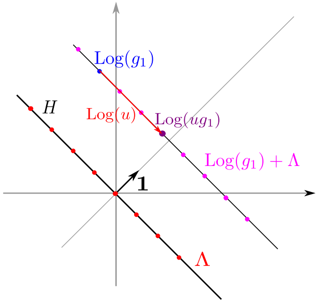

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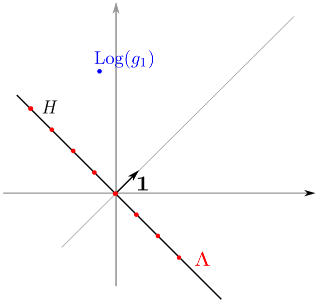

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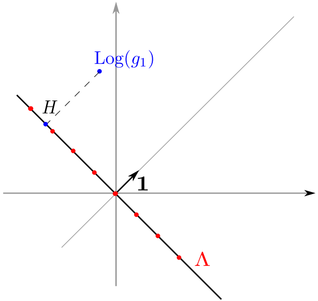

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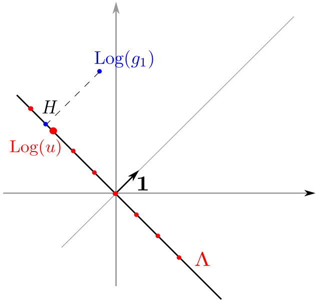

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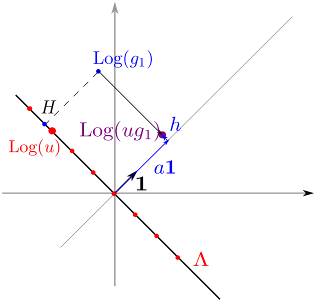

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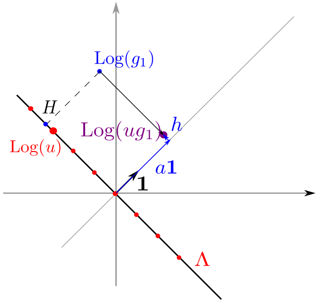

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- Solve CVP in Λ.
  - Good basis of Λ
    $\Rightarrow$ CVP in poly time
    $\Rightarrow \|h\| \leq \widetilde{O}(\sqrt{n})$

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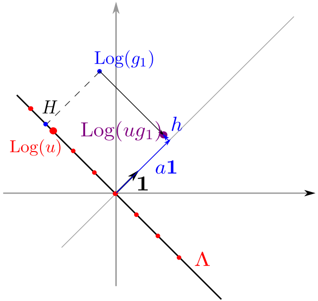

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    $\Rightarrow \|h\| \leq \widetilde{O}(\sqrt{n})$

$$\|ug_1\| \leq \mathcal{N}(ug_1)^{1/n} \cdot 2^{\widetilde{O}(\sqrt{n})}$$

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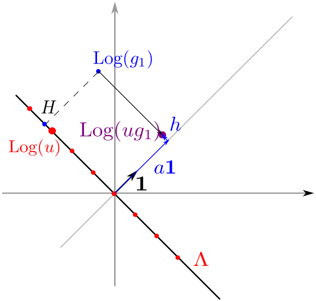
- $\|r\| \leq \sqrt{n} \cdot 2^a \cdot 2^{\|h\|}$
- $a = \frac{\log |\mathcal{N}(r)|}{n}$
- $\lambda_1 = \mathrm{poly}(n) \cdot \mathcal{N}(g)^{1/n}$



The CDPR Algorithm:

- Find a generator g_1 of $\langle g \rangle$.
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- Solve CVP in Λ.
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CDPR (upper bound)

Reminder ($\mathrm{Log}(r) = h + a\mathbf{1}$)

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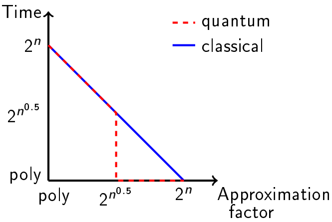

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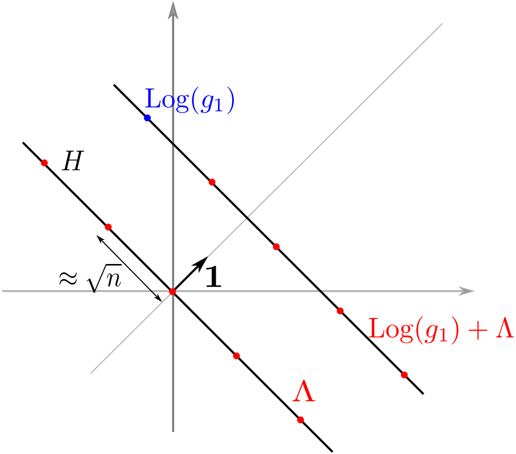
Outline of the talk

Idea



$\mathrm{Log}(g_1)$
H
$\approx \sqrt{n}$
$\mathbf{1}$
$\mathrm{Log}(g_1) + \Lambda$
Λ

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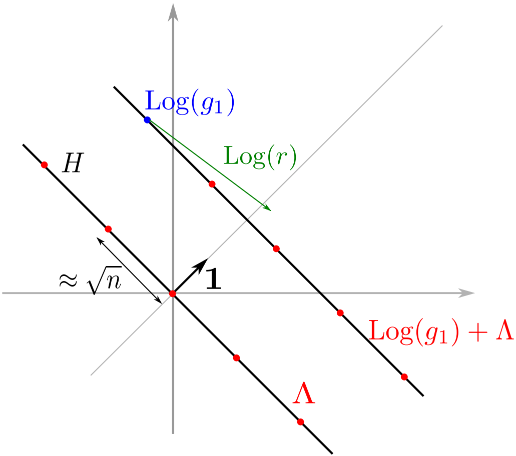

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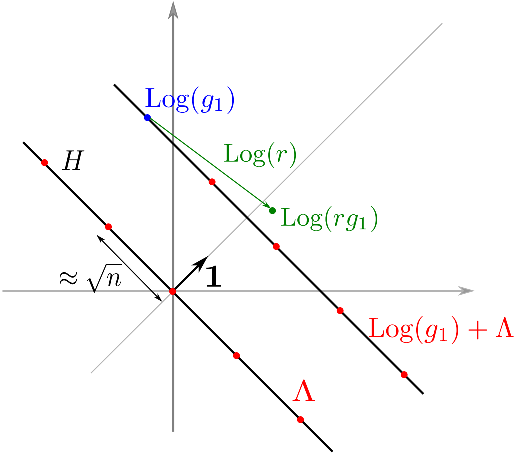

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$\mathrm{Log}(rg_1)$
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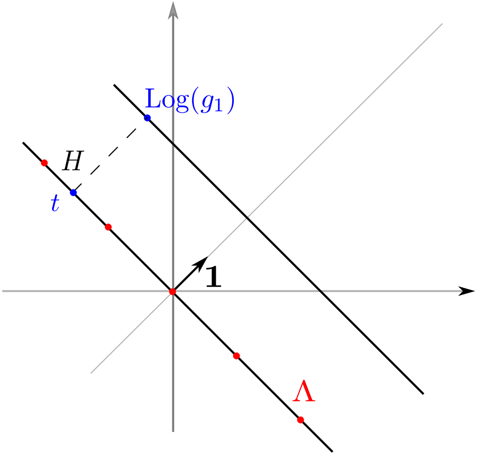

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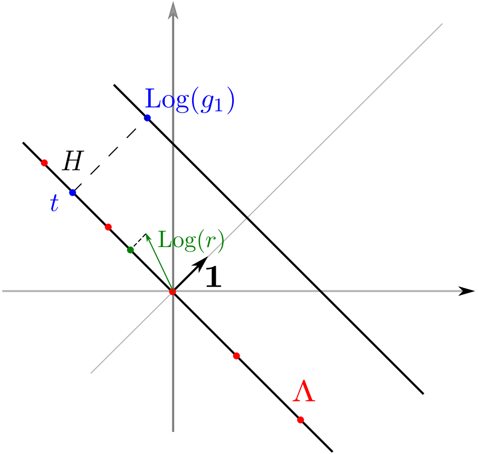

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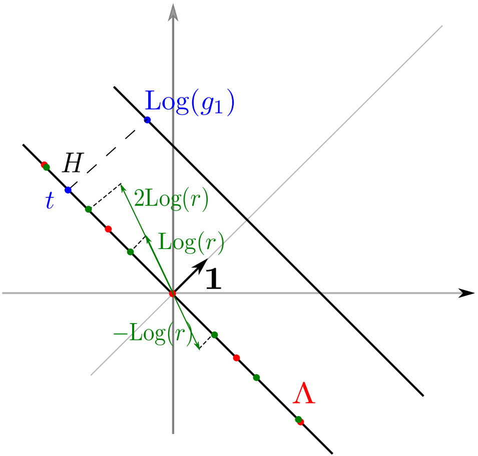

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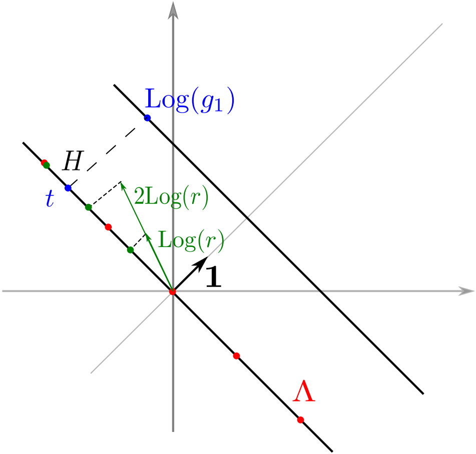

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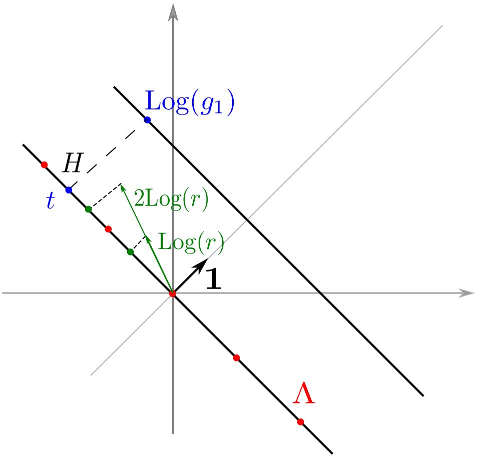

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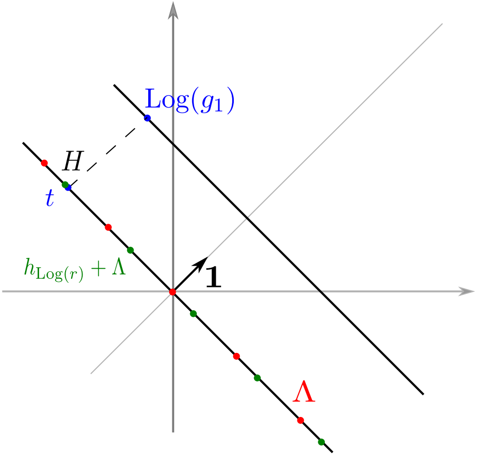

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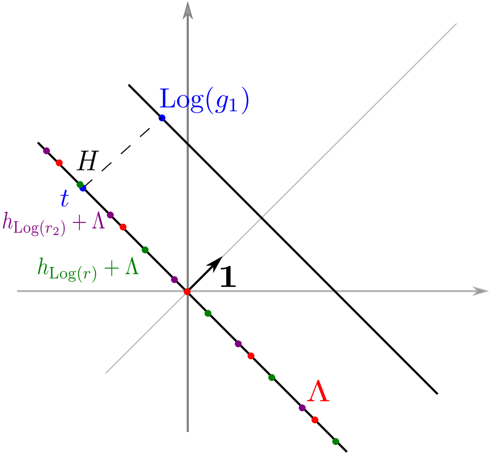
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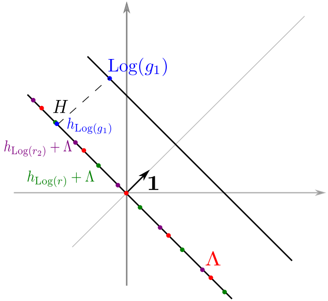
Λ

Formalisation

Difficulties

- We cannot subtract $\mathrm{Log}(r_i)$
- We cannot add too many $\mathrm{Log}(r_i)$'s

$\Rightarrow$ This is not a lattice



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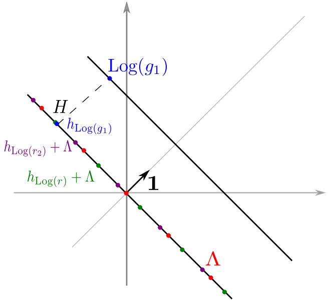

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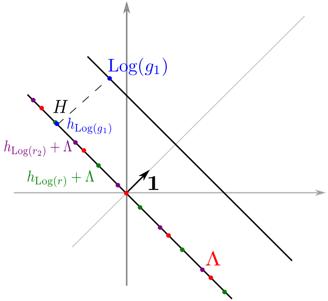

We consider the lattice and CVP target

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$-h_{\mathrm{Log}\, g_1}$
small > 0

Summary

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Compute $r_1, \cdots, r_n$ of small algebraic norms

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How to solve CVP in L?

CDPR	This work
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[Laa16]:
- Find $s \in L$ such that $\|s - t\| = \widetilde{O}(n^{\alpha})$
- Time: $2^{\widetilde{O}(n^{1-2\alpha})}$ (query)
  $+ \, 2^{O(n)}$ (pre-processing)

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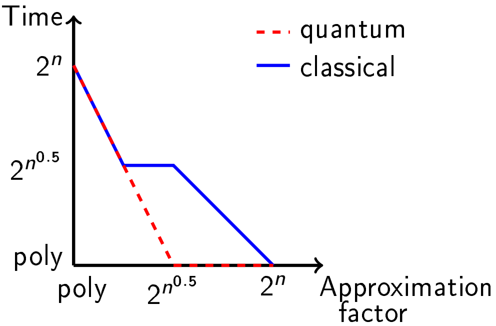
Conclusion

Approximation	Query time	Pre-processing
$2^{\widetilde{O}(n^{\alpha})}$	$2^{\widetilde{O}(n^{1-2\alpha})} + (\mathrm{poly}(n)$ or $2^{\widetilde{O}(\sqrt{n})})$	$2^{O(n)}$



$+2^{O(n)}$ Pre-processing / Non-uniform algorithm

Extensions

- Non principal ideals ✓
- Generalization to other number fields ?
- Removing (or testing) the heuristics ?

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Questions?