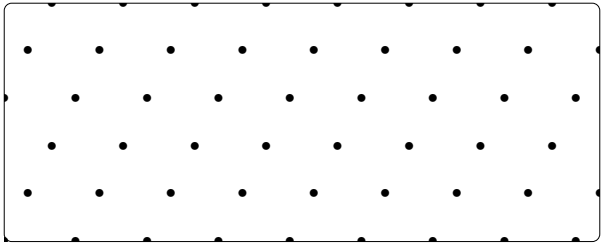
How to build cryptography from lattices

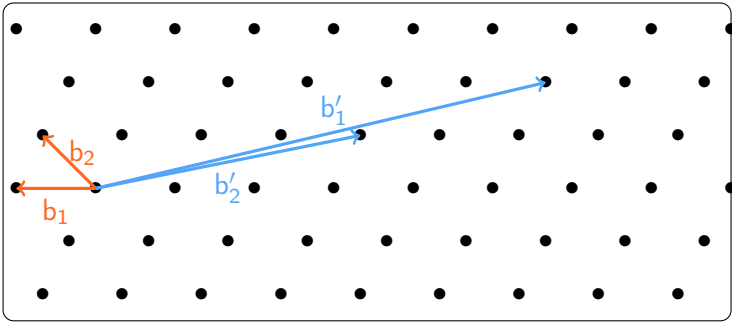
Alice Pellet-Mary

CAIPI Symposium, Limoges

Lattices and lattice problems



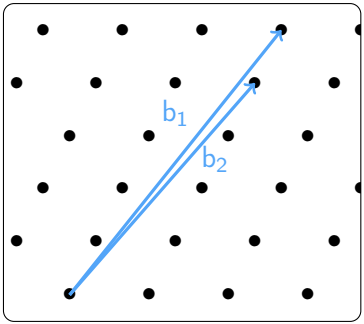
Lattices



- $\mathcal{L} = \{\sum_{i=1}^{n} x_i \mathbf{b}_i \mid \forall i,\ x_i \in \mathbb{Z}\}$ is a lattice
- $(\mathbf{b}_1, \ldots, \mathbf{b}_n) =: B \in \mathrm{GL}_n(\mathbb{R})$ is a basis (not unique)
- n is the dimension (or rank)

Short basis problem

Input:



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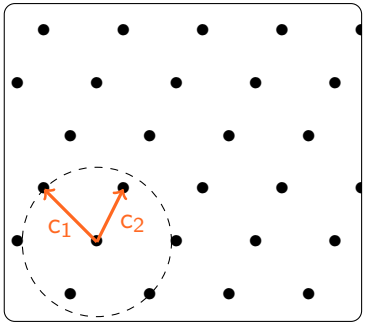
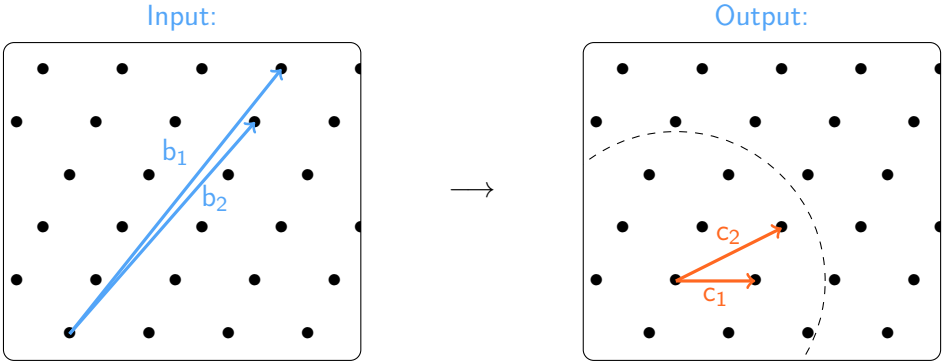


Shortest basis problem

$$\max_i \|c_i\| \leq \min_{B' \text{ basis of } \mathcal{L}} \left(\max_i \|b'_i\| \right)$$

Short basis problem

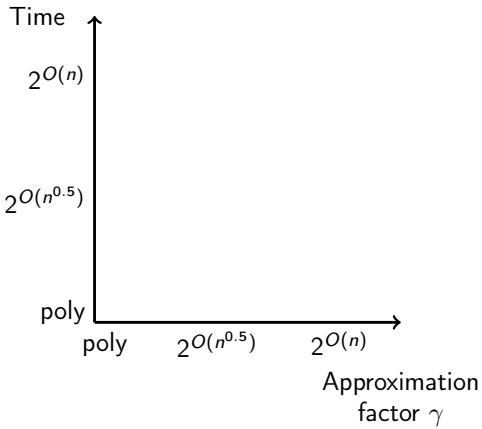


Approximate short basis problem

$$\max_i \|c_i\| \leq \gamma \cdot \min_{B' \text{ basis of } \mathcal{L}} \left(\max_i \|b'_i\| \right)$$

Asymptotic hardness

Best known asymptotic algorithms for the γ-short basis problem (classical and quantum)



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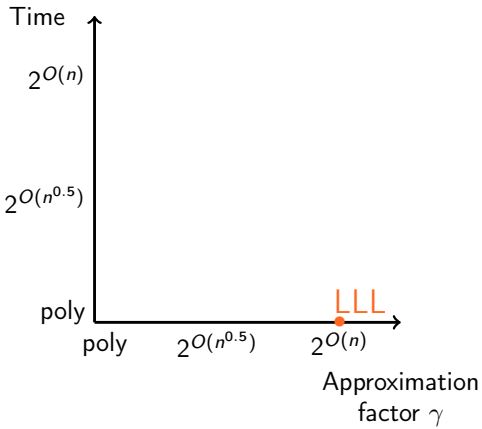

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- γ-short basis with $\gamma = 2^n$
- polynomial time

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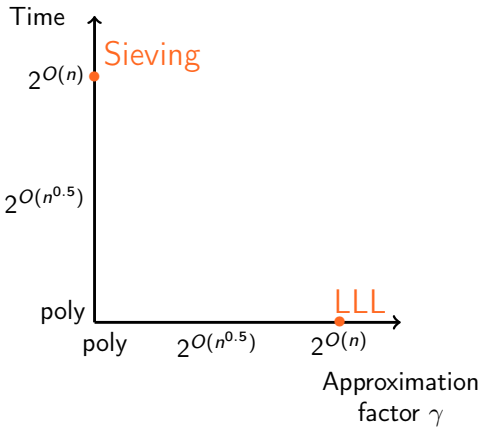

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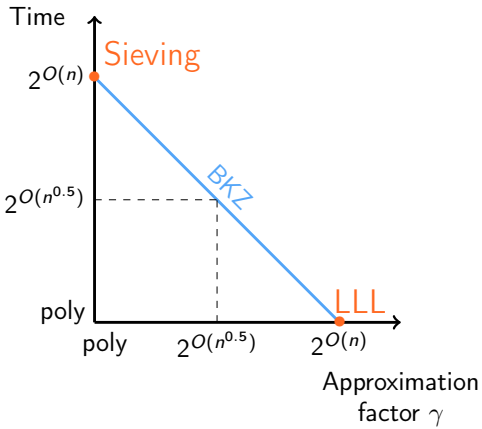

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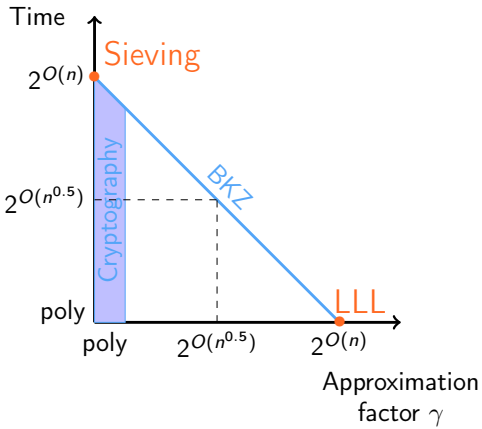

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- from $n = 400$ to $n = 1000$ $\rightsquigarrow$ cryptography

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Section's conclusion

Takeaway: computing short bases of lattices seems very hard (even quantumly)

- if n (the dimension) is large enough (e.g., $n \geq 1000$)
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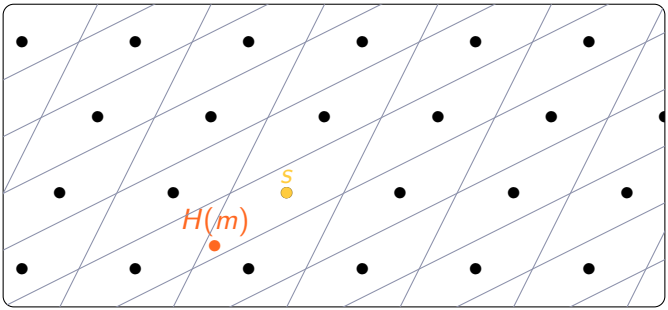
- For crypto we want to generate random hard lattices
  (We will come back to this...)

Constructing signatures from lattices



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Three algorithms:

- KeyGen() $\rightsquigarrow (pk, sk)$
- Sign$(m, sk) \rightsquigarrow s$
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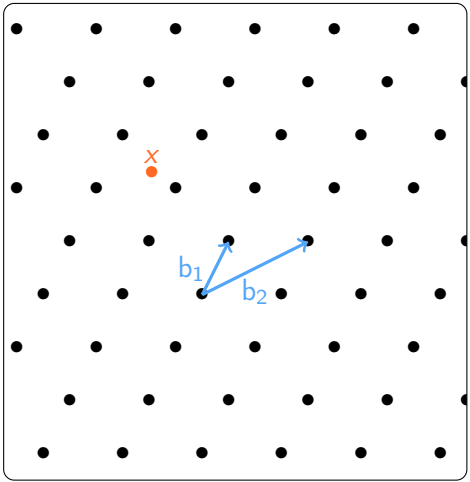
Security: an attacker not knowing sk cannot forge (m, s) s.t. $\mathrm{Verify}(m, s, pk) = \mathrm{yes}$

Finding a close vector using a short basis



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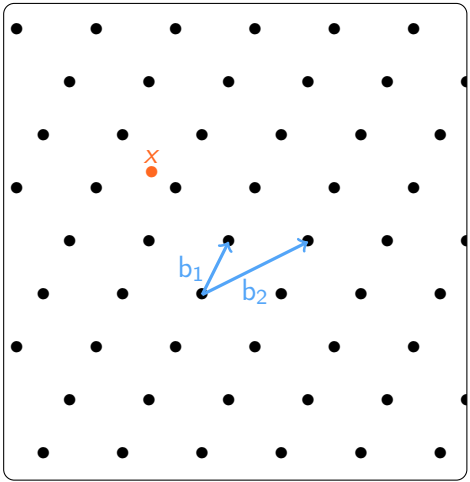

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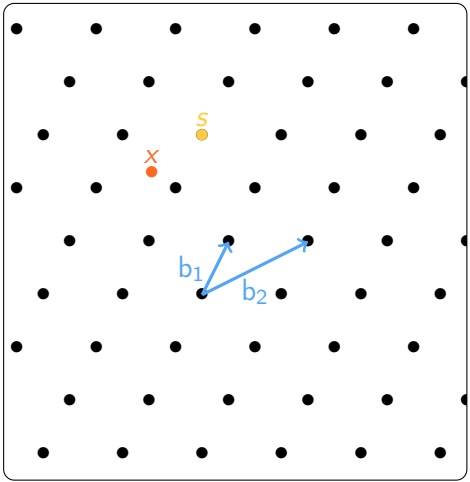

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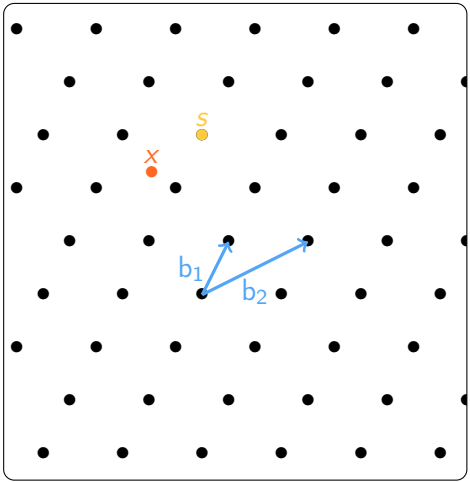

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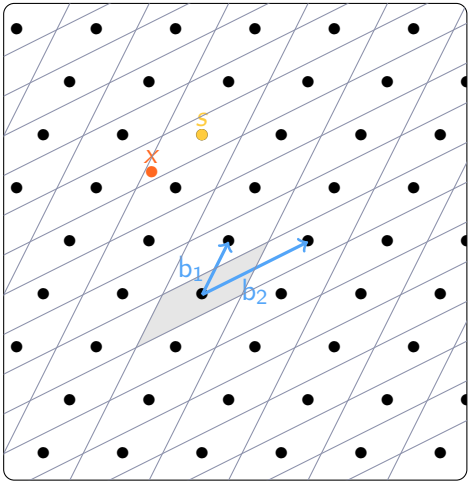

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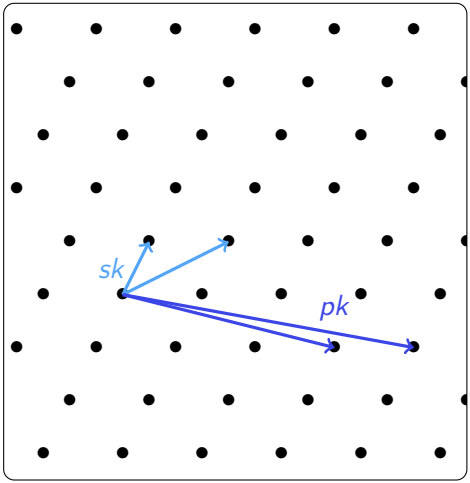
Hash-and-sign lattice signature [GGH97]



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- pk = bad basis of $\mathcal{L}$
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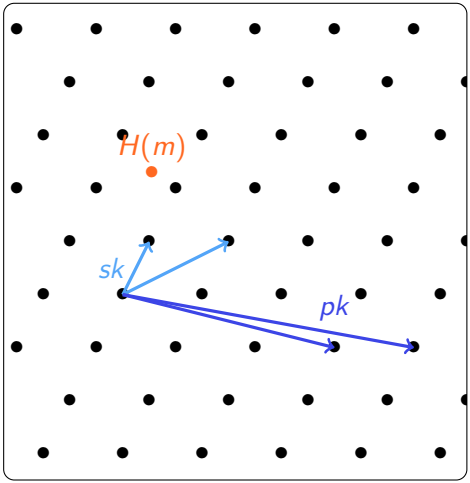

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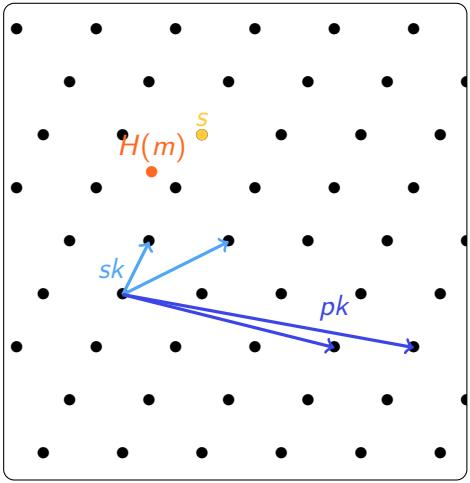

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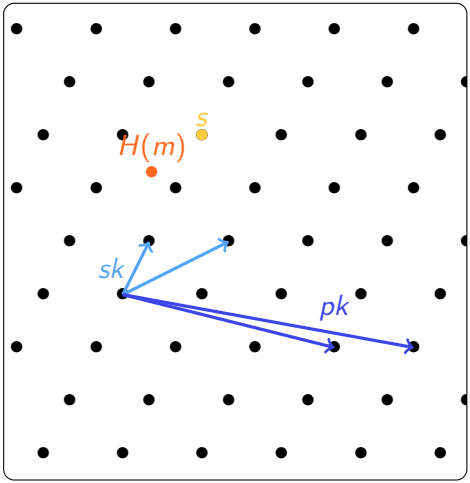

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Verify(m, s, pk):

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

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Warning: The scheme from previous slide is insecure [NR06] but it can be fixed with a small change [GPV08]

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Hash-and-sign signature scheme:

- requires a lattice $\mathcal{L}$ + a short basis $\mathbf{B}_s$ + a bad basis $\mathbf{B}_p$
- provably secure if recovering a short basis from $\mathbf{B}_p$ is hard

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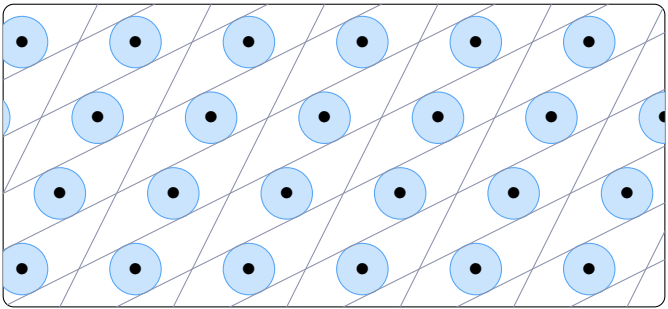
Constructing public key encryption from lattices



Public key encryption (PKE)

Three algorithms:

- $\mathsf{KeyGen}() \rightsquigarrow (pk, sk)$
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		message $m \in \{0,1\}$
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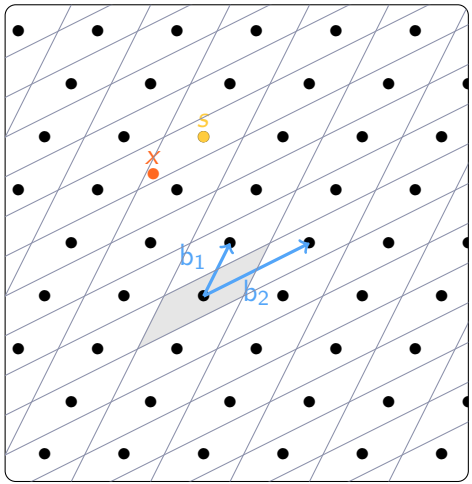
Security: an attacker cannot guess m from pk and $\mathrm{Enc}(m, pk)$

Remember Babai decoding



$\texttt{Babai}_B(x) = s \quad (\in \mathcal{L}, \text{ close to } x)$

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(fundamental parallelepiped)

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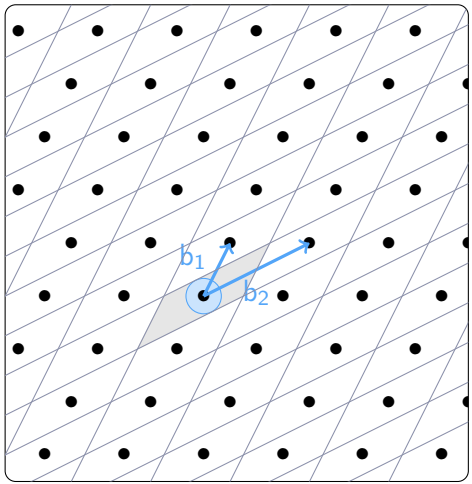

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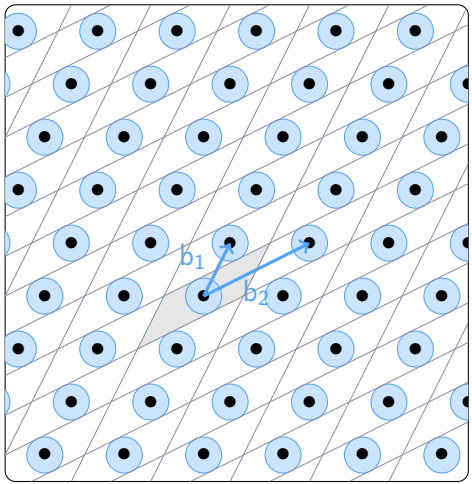

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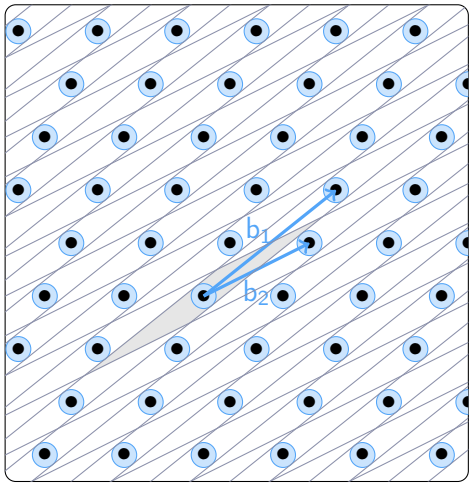

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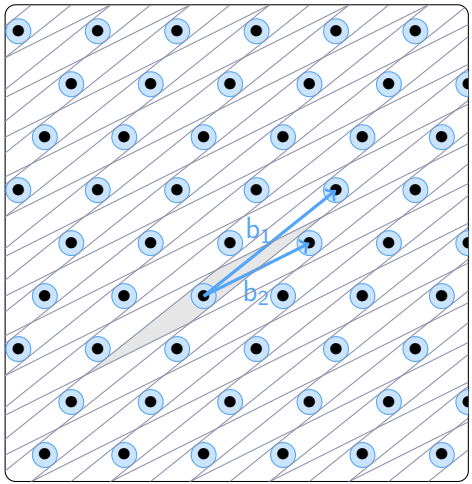

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Smaller basis $\iff$ larger ●

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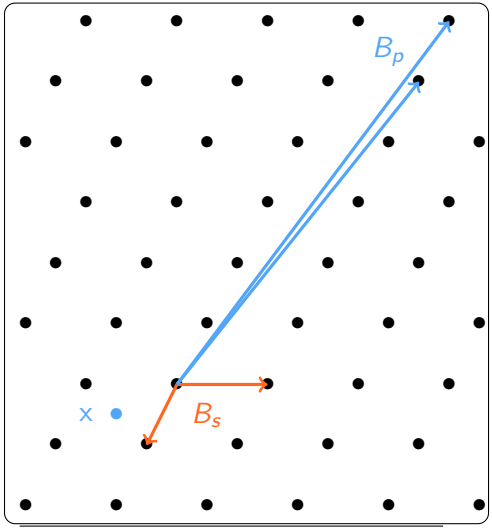

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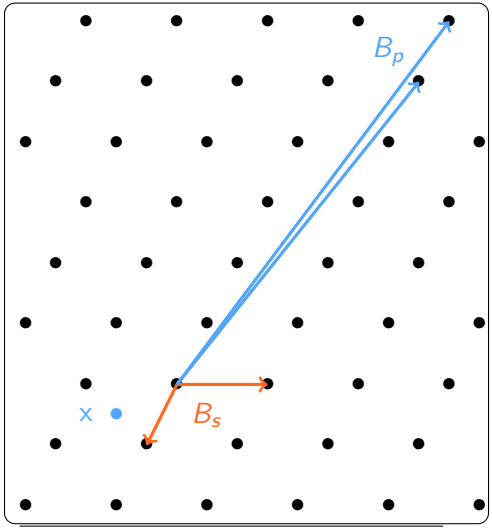

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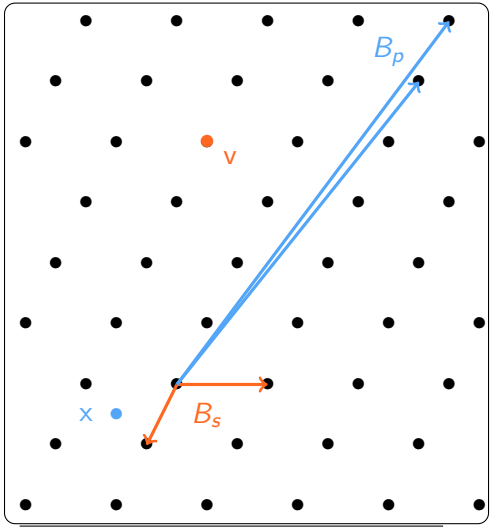

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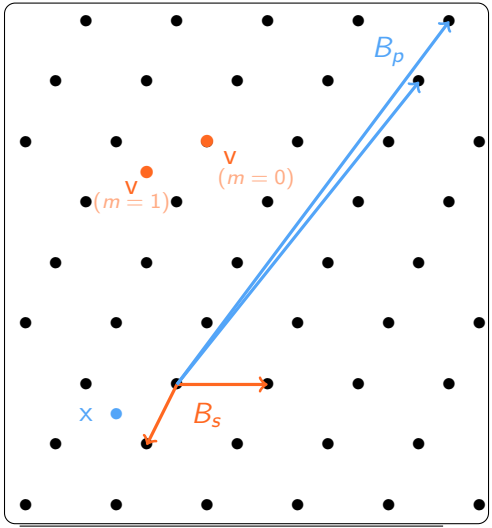

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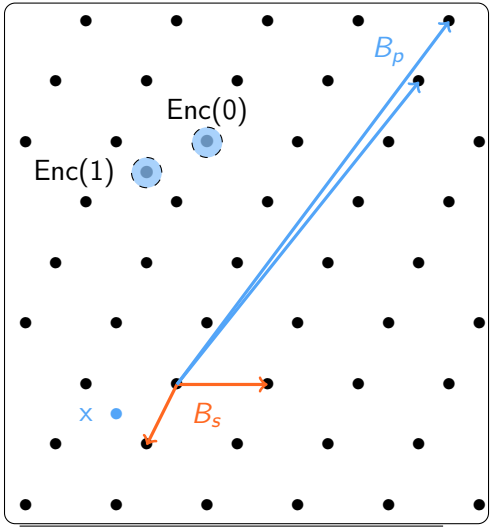

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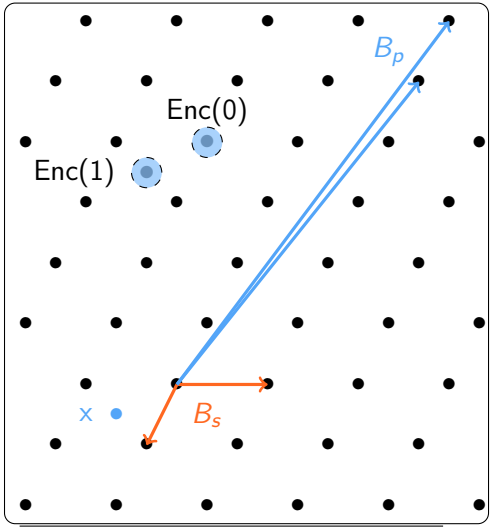

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Dec(c, sk):

- $\mathrm{s} \leftarrow \mathrm{Babai}_{sk}(\mathrm{c})$
- if $\|\mathrm{s} - \mathrm{c}\|$ small $\rightsquigarrow m = 0$
- else $\rightsquigarrow m = 1$

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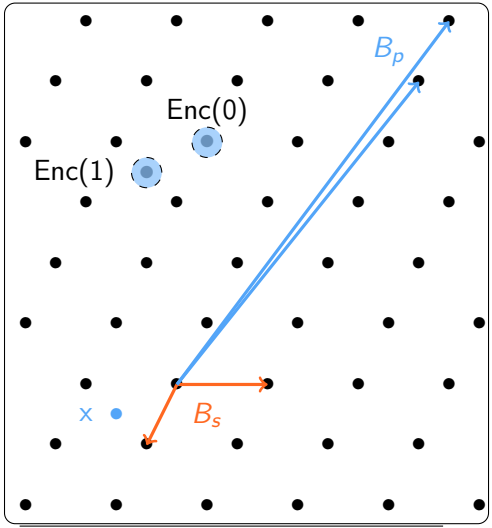

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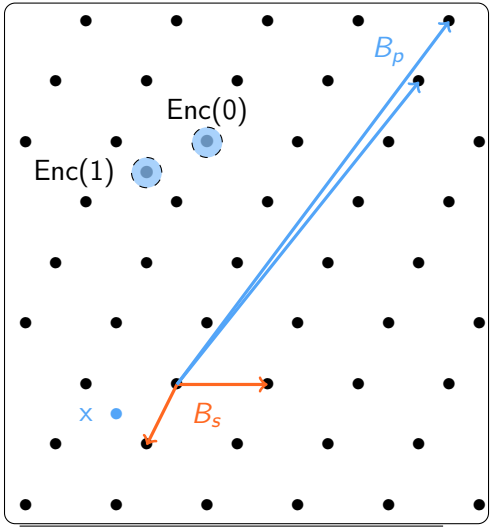

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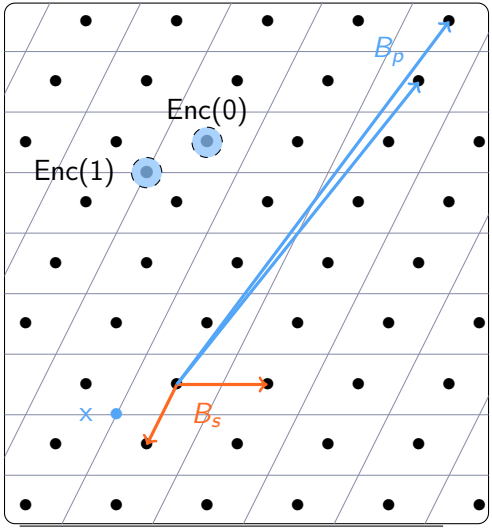

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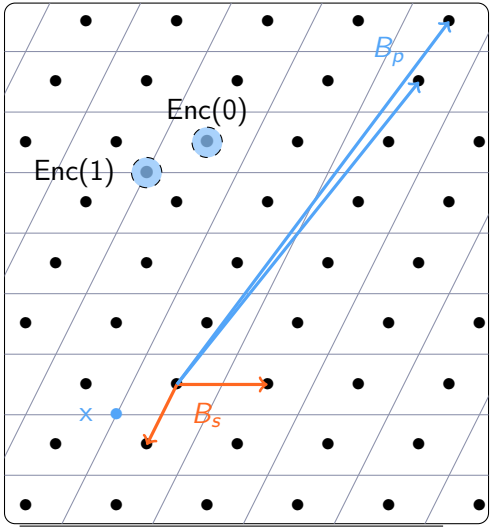

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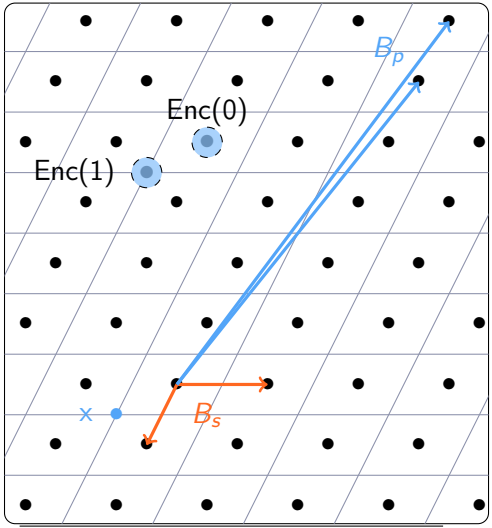

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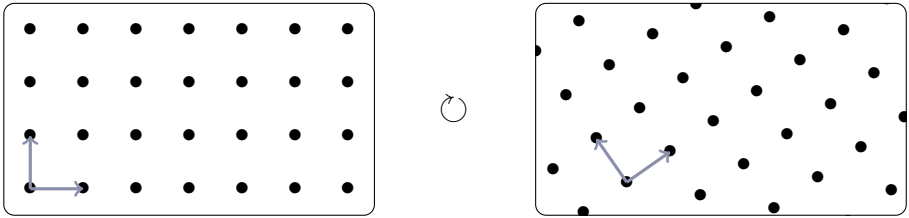
Security : ✓ if $\mathcal{L}$ is well chosen

[Reg05] Regev. On lattices, learning with errors, random linear codes, and cryptography. STOC.

Regev encryption:

- requires a lattice $\mathcal{L}$ + a short basis B_s + a bad basis B_p + a point x far from $\mathcal{L}$
- seems secure if finding close vectors given only B_p is hard

The lattice isomorphism problem



Objective

What we want: An algorithm `KeyGen` such that

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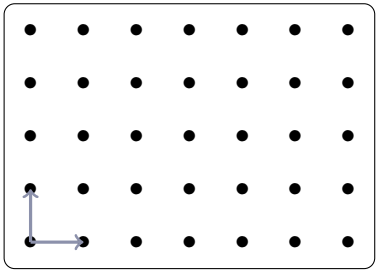
- computing a short basis of $\mathcal{L}$ from B_p is hard with overwhelming probability

The lattice isomorphism problem [DW22,BGPS23]



$$\mathcal{L}_0 = \mathbb{Z}^n$$

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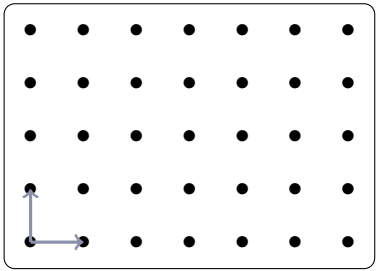

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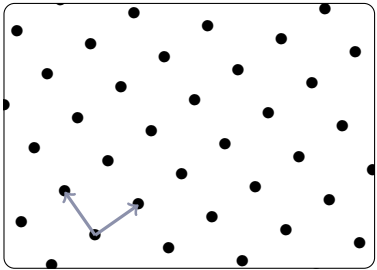

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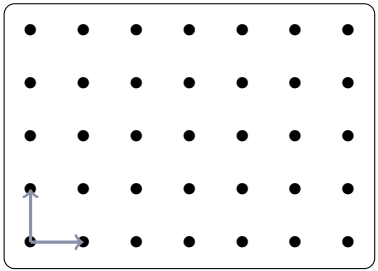

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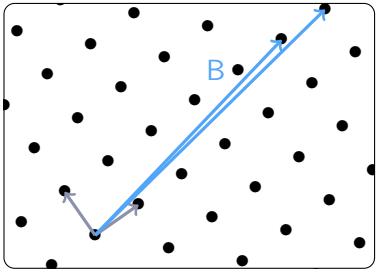

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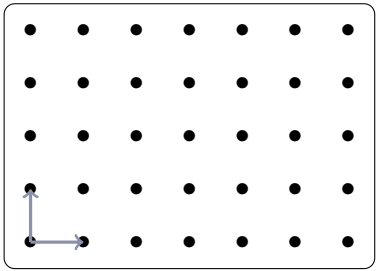

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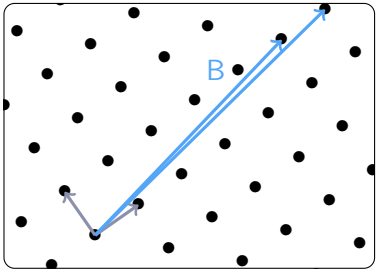

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Lattice Isomorphism Problem (LIP) assumption

recovering O from B is hard

$\Leftrightarrow$ computing a shortest basis of $\mathcal{L}$ is hard

Equivalent formulation with Gram matrices

Lattice isomorphism problem: Given $B = O \cdot C$ with

- $O \in O_n(\mathbb{R})$ orthogonal
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Find O (equivalently: find C)

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Lattice isomorphism problem (Gram matrix formulation):
Given $G = C^T C$ with C a (secret) basis of $\mathbb{Z}^n$, find C.

Section's conclusion

The lattice isomorphism problem (LIP):

- allows to generate random lattices
- where the short basis problem is believed to be hard to solve
- together with a short secret basis (and a point x far from the lattice)

We can use LIP to build public key encryption and signatures [DW22,BGPS23,DPPW23]

(There exists many other problems that provide (other distributions of) random hard lattices: LWE, SIS, module-LWE, module-SIS, NTRU, ...)

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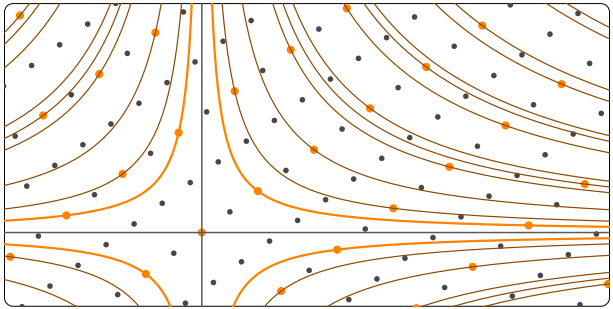
[DPPW23] Ducas, Postlethwaite, Pulles, van Woerden. Hawk: Module LIP makes lattice signatures Fast, Compact and Simple. Asiacrypt

Module lattices



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We extend $\sigma : K^k \to \mathbb{C}^{dk}$ coordinate-wise, and similarly $\|v\| := \|\sigma(v)\|_2$ for $v \in K^k$

(Free) module:

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$\sigma(M)$ is a lattice:

- with basis $(\sigma(b_i X^j))_{\substack{1 \leq i \leq k \\ 0 \leq j < d}}$ (b_i columns of B)
- of $\mathbb{Z}$-rank $n := d \cdot k$, included in $\mathbb{C}^n$

An example of module lattice

$K = \mathbb{Q}[X]/(X^2+2)$ $\qquad \sigma : K \to \mathbb{R}^2$

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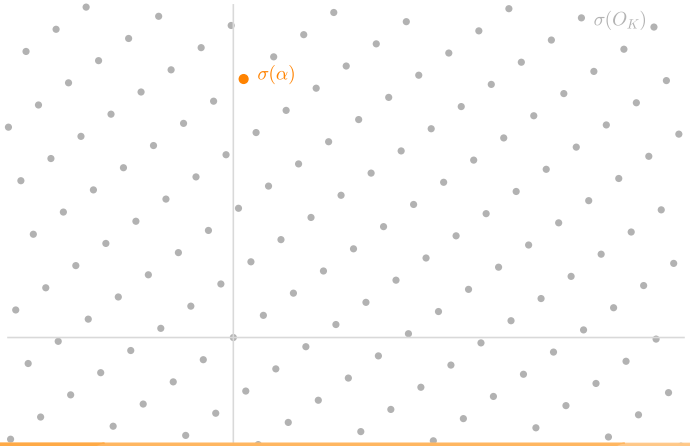

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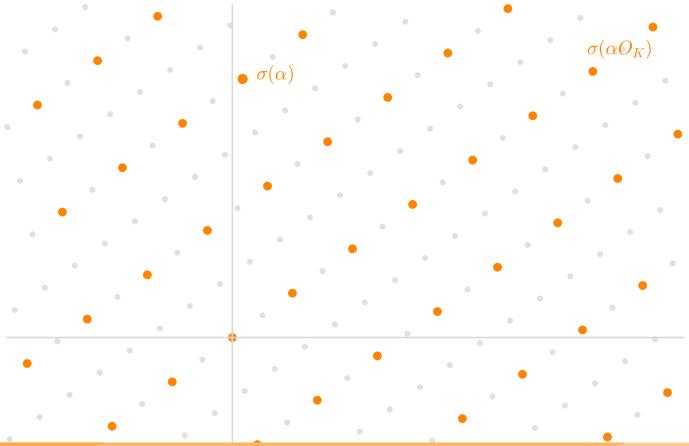

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Drawbacks:

- maybe more efficient attackers against some algorithmic problems?
  - for the short basis problem: better attacker if module rank $k = 1$, no better algorithm known when $k \geq 2$
  - for the lattice isomorphism problem: better attackers if module rank $k = 1$ and for some number fields when $k = 2$.

Wrapping up everything

To build crypto: we need random hard lattices
(lattices for which computing a short basis is hard)

The lattice isomorphism problem: allows to generate random hard lattices

Module lattices: add algebraic structure to improve efficiency of crypto constructions

What is the impact of the added algebraic structure on security?

I used beamer theme begles (https://framagit.org/squirrrr/beamerthemebegles)

Wrapping up everything

To build crypto: we need random hard lattices
(lattices for which computing a short basis is hard)

The lattice isomorphism problem: allows to generate random hard lattices

Module lattices: add algebraic structure to improve efficiency of crypto constructions

What is the impact of the added algebraic structure on security?

Thank you

I used beamer theme begles (https://framagit.org/squirrrr/beamerthemebegles)