Lattice-based crypto, part 1:
Algorithmic problems over lattices

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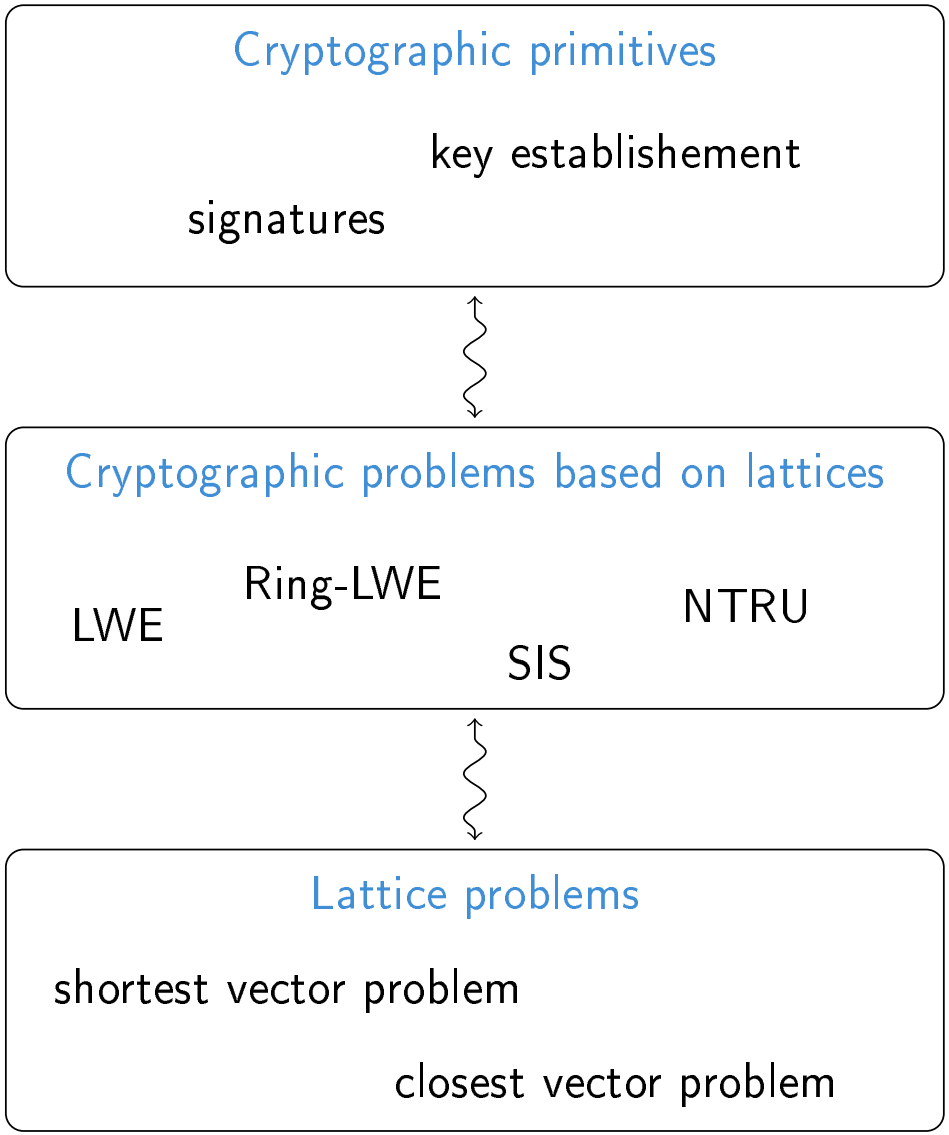
CNRS and university of Bordeaux, France

AsCrypto summer school
October 4-5, 2021

Plan of the talks



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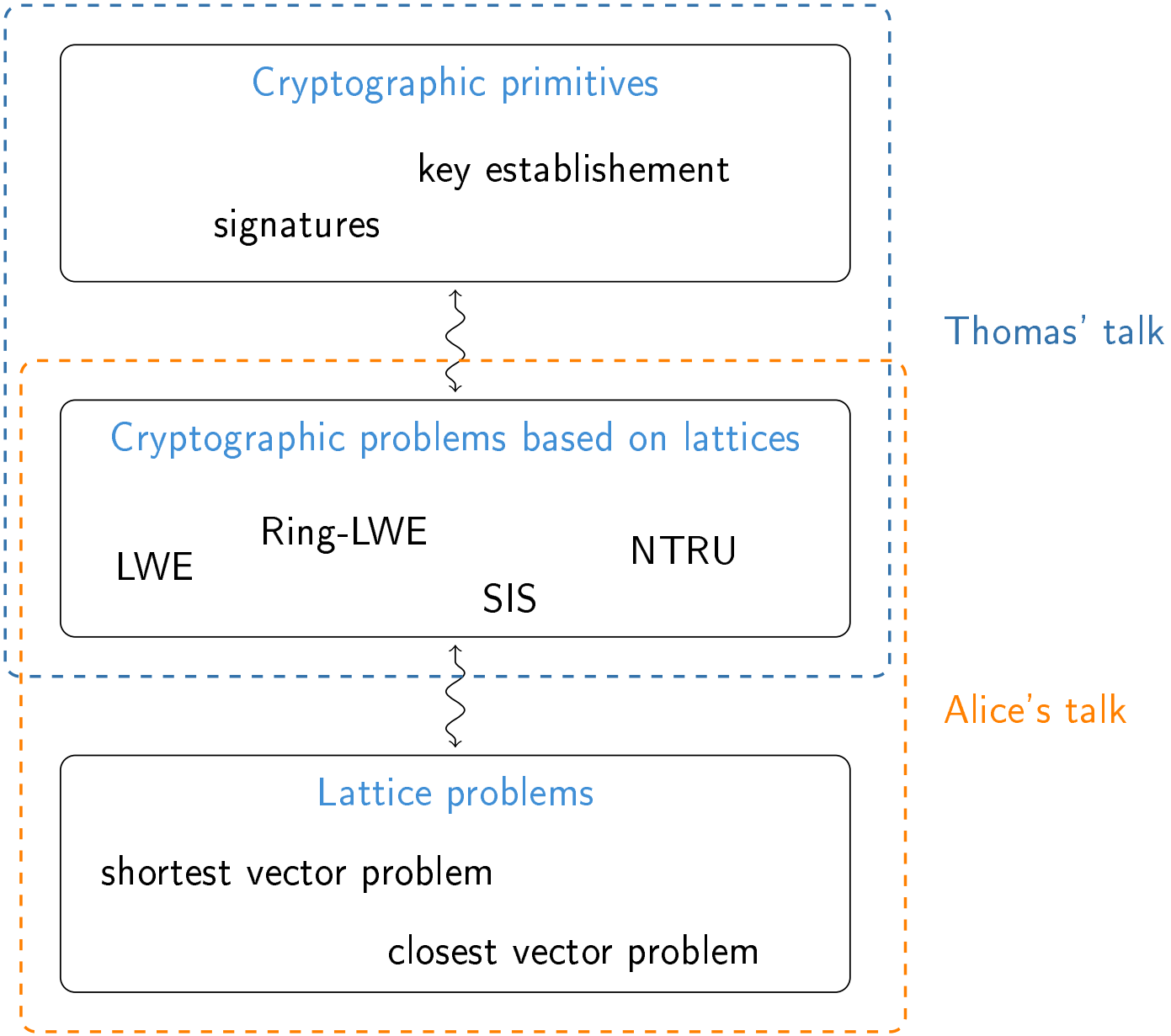

Outline of the talk

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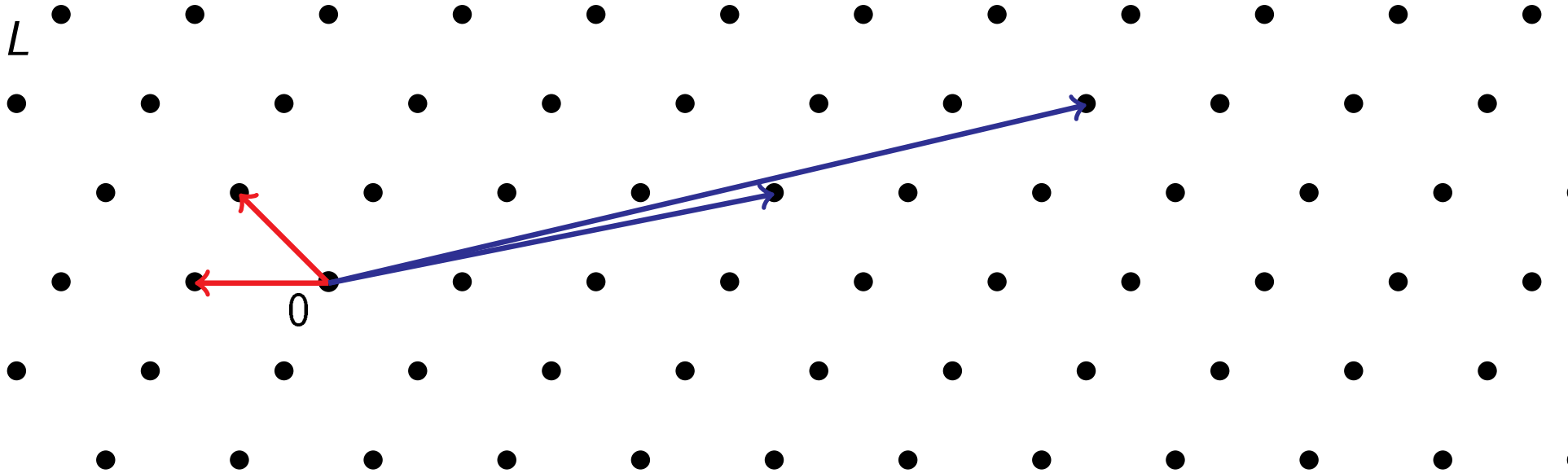
Lattices



- $L = \mathcal{L}(B) = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
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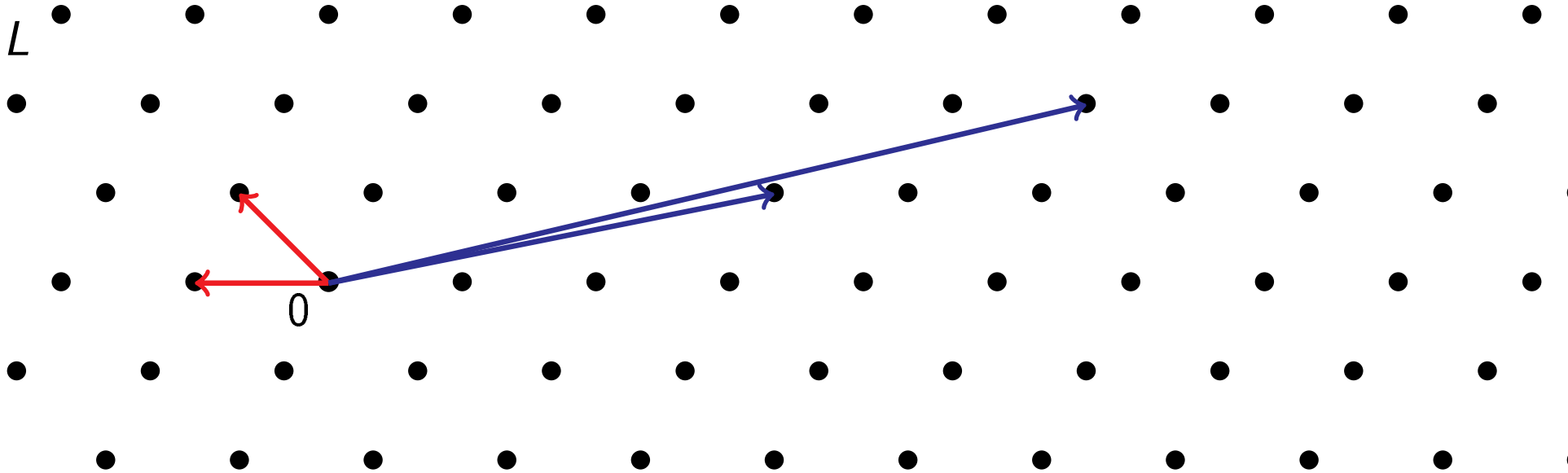
Lattices



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We represent a lattice by any of its basis

Algorithmic problems on lattices

Input: any basis of any lattice

Example of problems:

(1) Testing equality of lattices

(2) Testing inclusion of lattices

(3) Intersecting two lattices

(4) Computing a short vector of a lattice

(5) Computing a lattice vector close to a target

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Quiz: which ones are easy or hard?

easy: polynomial time hard: no polynomial time algorithm known

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(1) Testing equality of lattices $\Rightarrow$ easy

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(4) Computing a short vector of a lattice $\Rightarrow$ hard

(5) Computing a lattice vector close to a target $\Rightarrow$ hard

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Testing inclusion / equality

Exercise

Given $B_1, B_2 \in \mathrm{GL}_n(\mathbb{R})$, how do you test if

1. $\mathcal{L}(B_1) \subseteq \mathcal{L}(B_2)$
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Solution:

1.

$$\mathcal{L}(B_1) \subseteq \mathcal{L}(B_2) \Leftrightarrow B_1 = B_2 \cdot X \quad \text{for some } X \in \mathbb{Z}^{n \times n}$$
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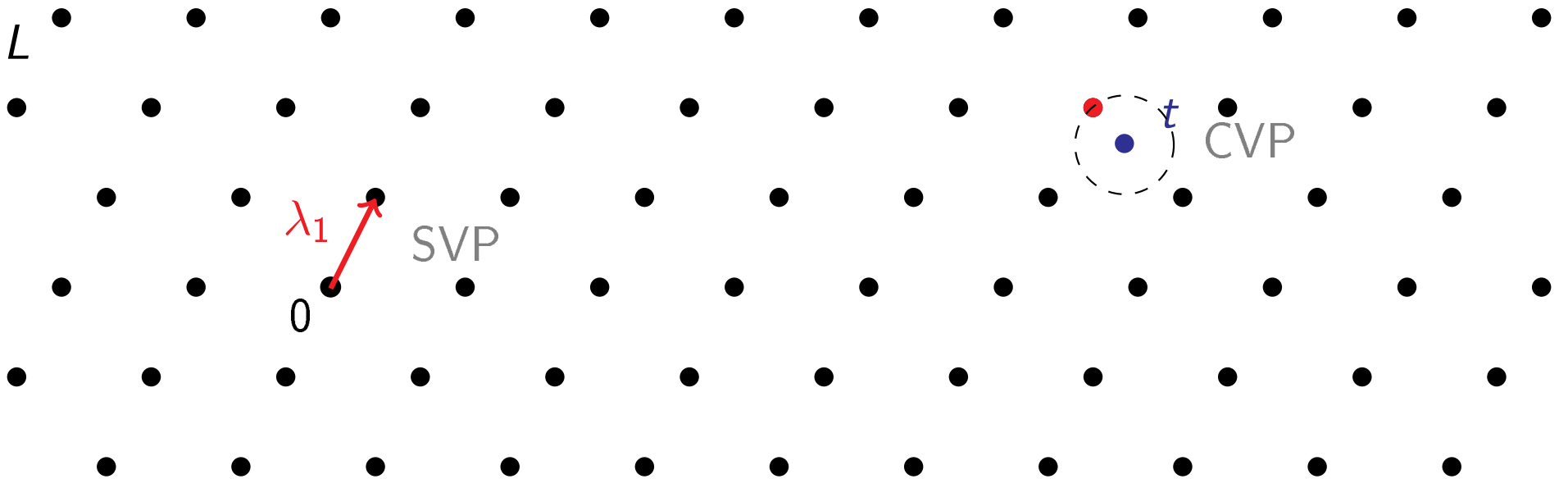
2. $\mathcal{L}(B_1) = \mathcal{L}(B_2) \Leftrightarrow \mathcal{L}(B_1) \subseteq \mathcal{L}(B_2)$ and $\mathcal{L}(B_2) \subseteq \mathcal{L}(B_1)$

(Hard) Lattice problems



SVP : Shortest Vector Problem

CVP : Closest Vector Problem

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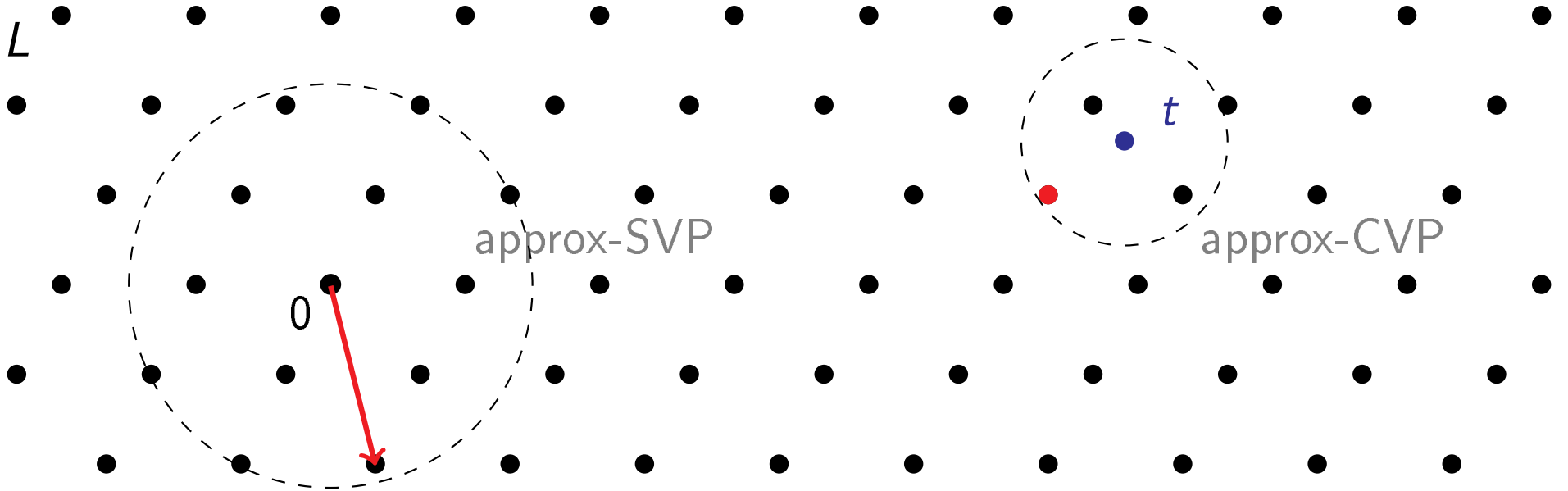

approx-SVP : Shortest Vector Problem

approx-CVP : Closest Vector Problem

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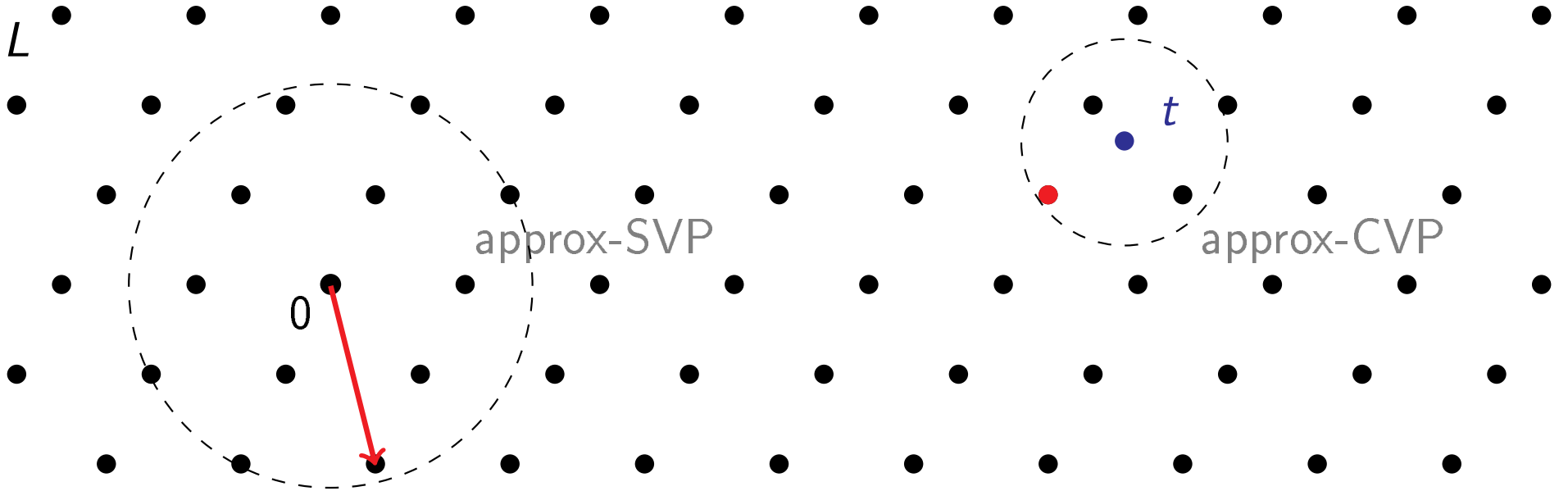

approx-SVP : Shortest Vector Problem approx-CVP : Closest Vector Problem

Supposedly hard to solve when n is large (input: a bad basis of L)

- even with a quantum computer
- even with a small approximation factor $(\mathrm{poly}(n))$

SVP and CVP in dimension 2 – Exercise

Exercise

Let $B = \begin{pmatrix} 13 & 10 \\ 2 & 2 \end{pmatrix}$ and $t = \begin{pmatrix} 20 \\ -1 \end{pmatrix}$

1. Solve SVP in $\mathcal{L}(B)$

2. Solve CVP in $\mathcal{L}(B)$ with target t

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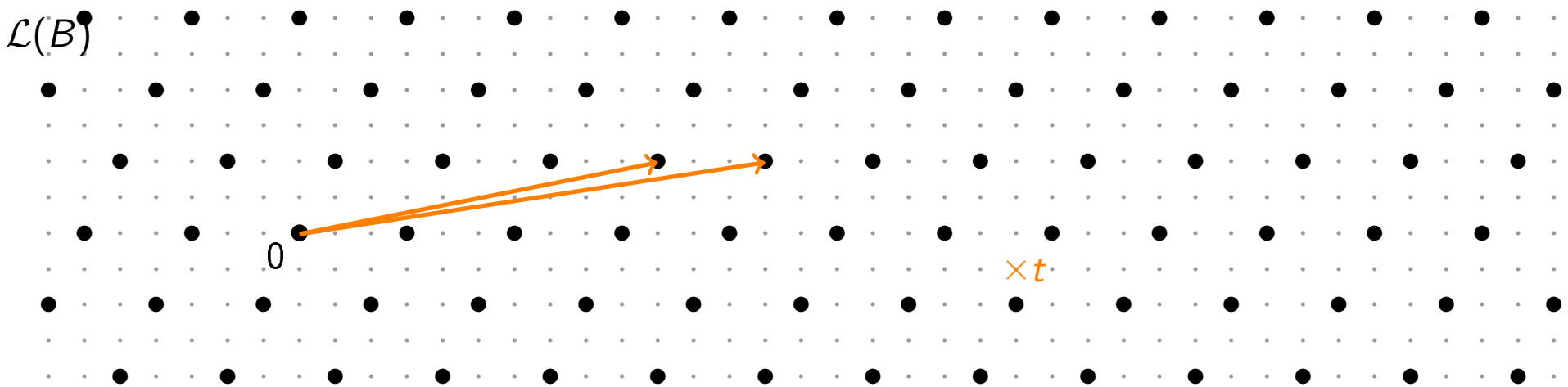

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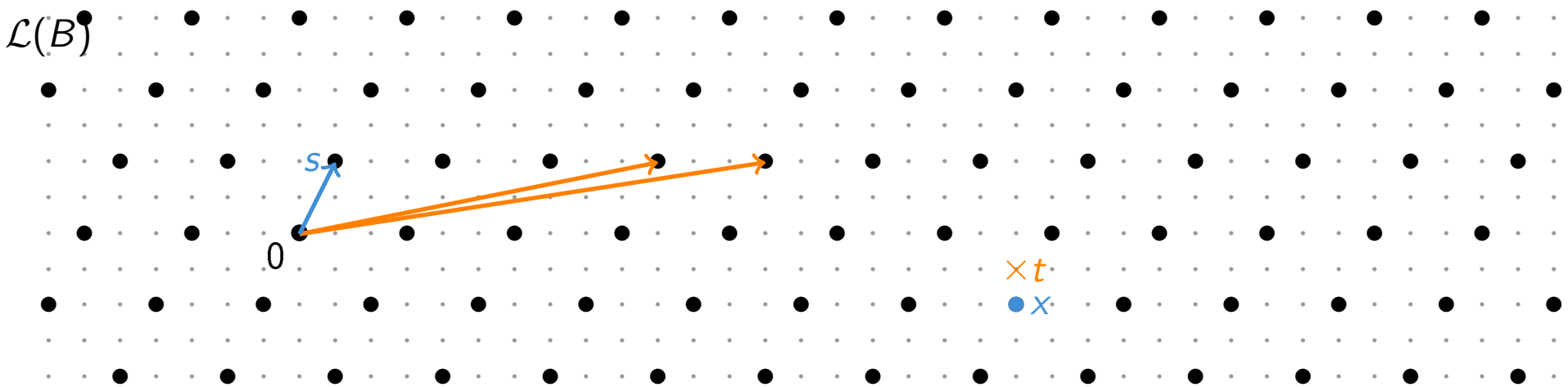
1. Solve SVP in $\mathcal{L}(B) \Rightarrow s = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
2. Solve CVP in $\mathcal{L}(B)$ with target $t \Rightarrow x = \begin{pmatrix} 20 \\ -2 \end{pmatrix}$



Algorithm for solving SVP in dimension 2

The Lagrange-Gauss algorithm:

- For lattices of rank $n = 2$ only
- Solves exact SVP
- Polynomial time

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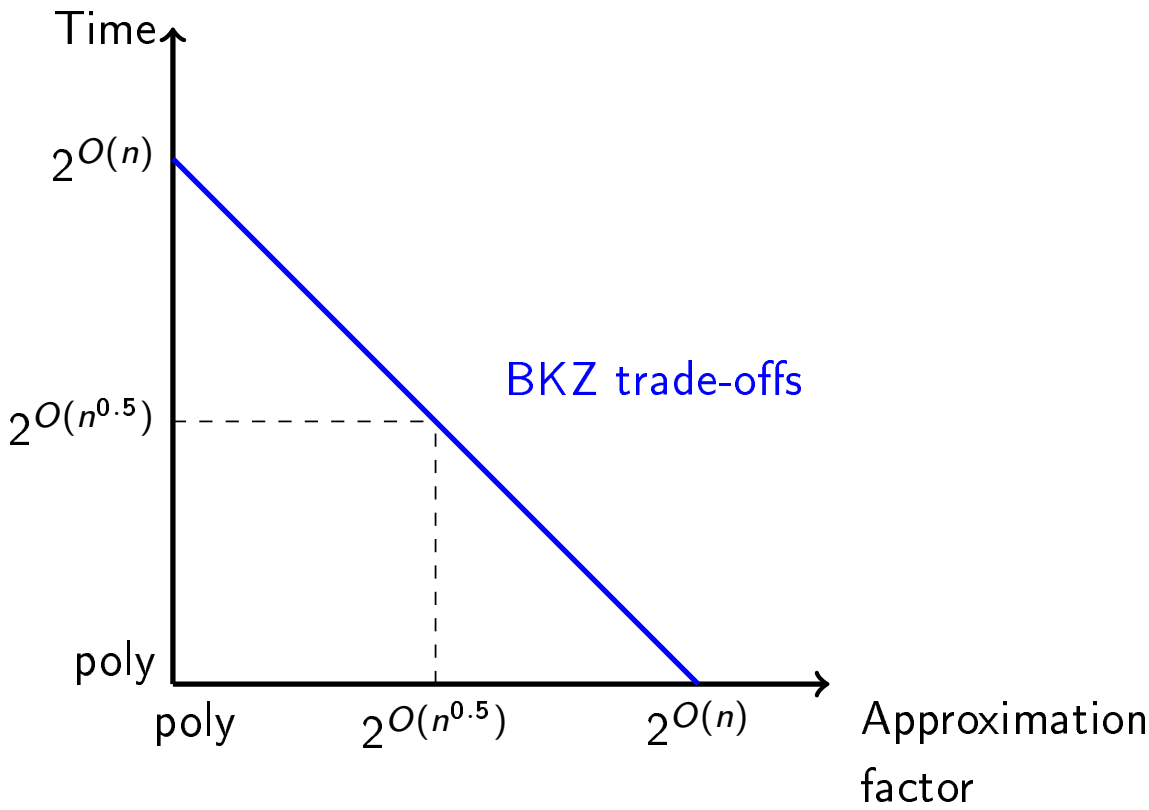
But remember: when n is large, solving exact SVP is hard

Asymptotic hardness of SVP and CVP

Best Time/Approximation trade-off for SVP, CVP (even quantumly):
BKZ algorithm [Sch87,SE94]



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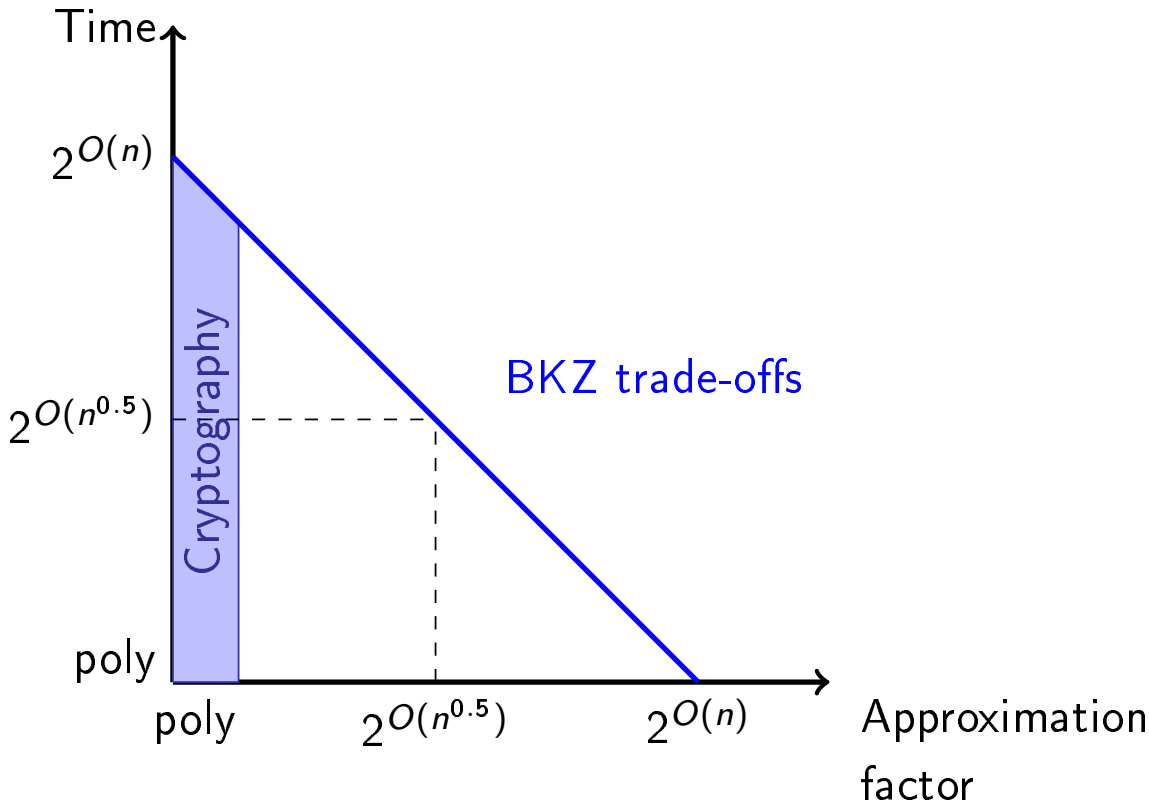

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Exercise

Do exercise 1 of the exercises sheet (`https://apelletm.pages.math.cnrs.fr/page-perso/documents/presentations/AsCrypto_exercises.pdf`).

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Solution: in my case, it was $\texttt{dim} \approx 57$

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Limitations of SVP (and CVP)

SVP and CVP are hard in the worst case

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  - Solution: $\texttt{dim} \approx 170$

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- no efficient algorithm that works for any lattice
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  - Exercise 2
  - Solution: $\texttt{dim} \approx 170$

For crypto, we need problems that are hard on average

(i.e., for a random instance, the problem is hard with overwhelming probability)

The SIS problem

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

SIS (Short Integer Solution) [Ajt96]

Given $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{m \times n})$ (with $n \log q < m$)

Find $x \in \{-B, \cdots, B\}^m \setminus \{0\}$ s.t. $x \cdot A = 0 \bmod q$

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Solving SIS with non-negligible probability (e.g., $\geq 2^{-80}$) $\overset{\sim}{\iff}$ Solving SVP in any lattice of rank n

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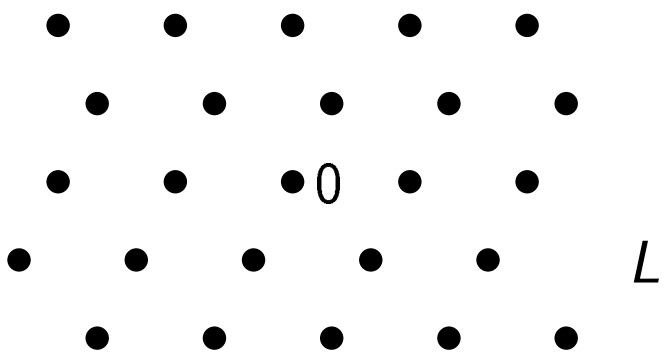

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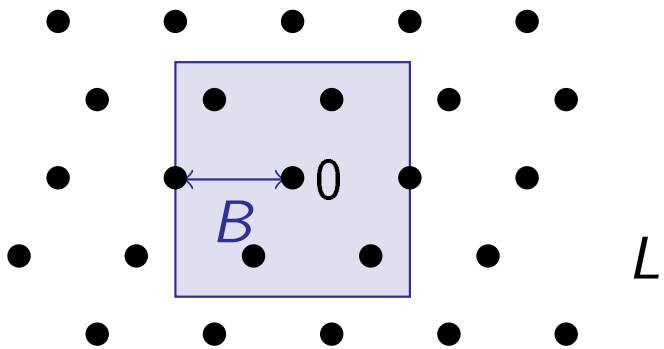

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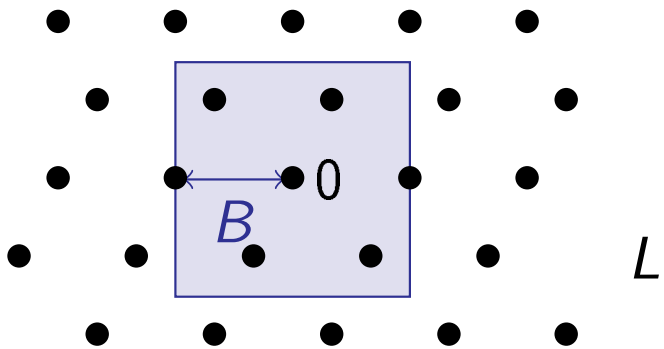

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Exercise 3 (worksheet)

Solution of exercise 3

```
set_random_seed(42)
A = random_matrix(Integers(127),5,10)

# kernel of A modulo q
B1 = Matrix(A.right_kernel().basis())
# all the (0,q,0,...,0) vectors
B2 = 127*identity_matrix(10)
# concatenation of the rows
B = Matrix(ZZ, list(B1)+list(B2))
print(B)

x = IntegerLattice(B).shortest_vector()
print(x)
print(A*x)
```

$$x = (1, 0, -1, 5, 5, 4, -2, 3, -3, -1)$$

The LWE problem

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LWE (Learning With Errors) [Reg05]

Sample $\mathbf{A} \leftarrow \text{Uniform}(\mathbb{Z}_q^{m \times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \text{Uniform}(\{-B, \cdots, B\}^n)$

Given $\mathbf{A}$ and $\mathbf{b}$, where $\mathbf{b} := \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$

Recover s or e

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Remark. Sometimes $\mathbf{s}$ is uniform in $\mathbb{Z}_q$ (not small)

- this is (almost) equivalent
- prove it (*hint:* you are allowed to change m)

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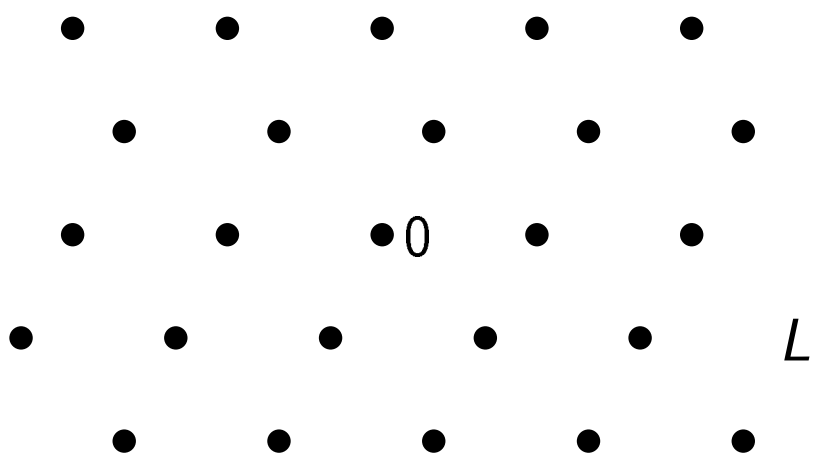

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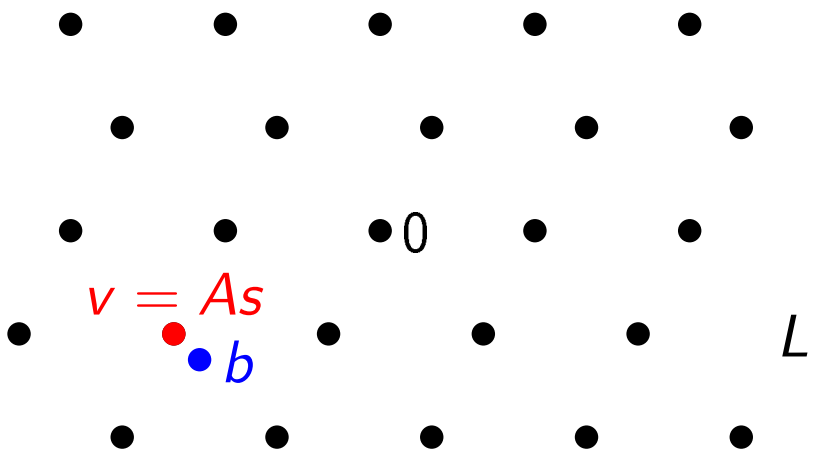

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$$b = v + e,$$

where $v \in L$ and e small

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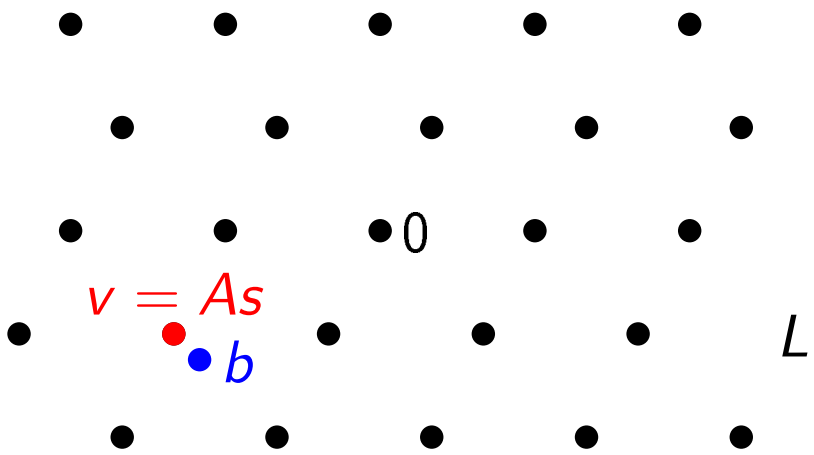

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$\Rightarrow$ Good for crypto

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SIS $\overset{\sim}{\longleftrightarrow}$ average case SVP

LWE $\overset{\sim}{\longleftrightarrow}$ average case CVP

Decision variant of LWE

decision-LWE

Sample $A \leftarrow \mathrm{Uniform}(\mathbb{Z}_q^{n \times n})$ and $s, e \leftarrow \mathrm{Uniform}(\{-B, \cdots, B\}^n)$

Given A and b, where

$$b := A s + e \bmod q \quad \text{or} \quad b \leftarrow \mathrm{Uniform}(\mathbb{Z}_q^n)$$

Guess whether b is uniform or not.

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$\Rightarrow$ decision problems can be easier to use for crypto

LWE vs SIS

$$\text{decision-LWE} \overset{\sim}{\Longleftrightarrow} \text{(search) LWE} \overset{\sim}{\Longleftrightarrow} \text{SIS}$$

LWE vs SIS

$$\text{decision-LWE} \overset{\sim}{\Longleftrightarrow} \text{(search) LWE} \overset{\sim}{\Longleftrightarrow} \text{SIS}$$

Exercise

Prove that decision-LWE $\leq$ SIS

Hint: Assume that we know $\mathbf{x}$ small such that $\mathbf{x}\,\mathbf{A} = 0 \bmod q$

How to distinguish

$$\mathbf{b} := \mathbf{A}\,\mathbf{s} + \mathbf{e} \bmod q \quad \text{vs} \quad \mathbf{b} \leftarrow \text{Uniform}(\mathbb{Z}_q^n)$$

LWE vs SIS

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Hint: Assume that we know $\mathbf{x}$ small such that $\mathbf{x}\,\mathbf{A} = 0 \bmod q$

How to distinguish

$$\mathbf{b} := \mathbf{A}\,\mathbf{s} + \mathbf{e} \bmod q \quad \text{vs} \quad \mathbf{b} \leftarrow \text{Uniform}(\mathbb{Z}_q^n)$$

Solution: Compute $\mathbf{x}\,\mathbf{b}$, this gives

$$\underbrace{\mathbf{x}\,\mathbf{A}}_{0}\,\mathbf{s} + \mathbf{x}\,\mathbf{e} = \underbrace{\mathbf{x}\,\mathbf{e}}_{\text{small}} \bmod q \quad \text{vs} \quad \underbrace{\mathbf{x}\,\mathbf{b}}_{\text{uniform}}$$

Outline of the talk

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$$R = \mathbb{Z}[X]/(X^n + 1) \qquad \text{with } n = 2^k$$

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$n = 4$, $r_1 = 1 + 3X - X^3$ and $r_2 = 2X + 4X^2 + X^3$

$r_1 + r_2 = ??$ $\qquad$ $r_1 \cdot r_2 = ??$

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$r_1 + r_2 = 1 + 5X + 4X^2 \qquad r_1 \cdot r_2 = -1 + 6X + 11X^2 + 13X^3$

Ring-LWE (RLWE)

$R_q := R/(qR) = \mathbb{Z}_q[X]/(X^n + 1)$

decision-RLWE [SSTX09, LPR10]

Sample

- $a \leftarrow \text{Uniform}(R_q)$
- $s, e \in R$ with coefficients in $\{-B, \cdots, B\}$

Distinguish between

$$b = a \cdot s + e \bmod q \quad \text{and} \quad b \leftarrow \text{Uniform}(R_q)$$

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Solution: $M_a = \begin{pmatrix} a_0 & -a_{n-1} & \cdots & -a_2 \\ a_1 & a_0 & \cdots & -a_3 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_0 \end{pmatrix}$

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Conclusion (so far): better efficiency for the same security guarantees

NTRU

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NTRU [HPS98]

Sample $f, g \in R$ with coefficients in $\{-B, \cdots, B\}$.
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$$h = f \cdot g^{-1} \bmod q \quad \text{and} \quad h \leftarrow \text{Uniform}(R_q)$$

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1. check that $f \cdot h = g \bmod q$
2. why is it unsafe to take $h = f$ or $h = g^{-1} \bmod q$?
   - f is small, easy to distinguish from Uniform(R_q) (which is likely $\approx q$)
   - if $h = g^{-1}$, one can compute $h^{-1} = g \bmod q$ and same situation as above

Hardness of NTRU

- NTRU is assumed to be post-quantumly hard when n is large and $q = \mathrm{poly}(n)$

[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

Hardness of NTRU

- NTRU is assumed to be post-quantumly hard when n is large and $q = \mathrm{poly}(n)$
- NTRU $\leq$ RLWE [Pei16]
- we do not know whether NTRU $\approx$ RLWE or NTRU $<$ RLWE

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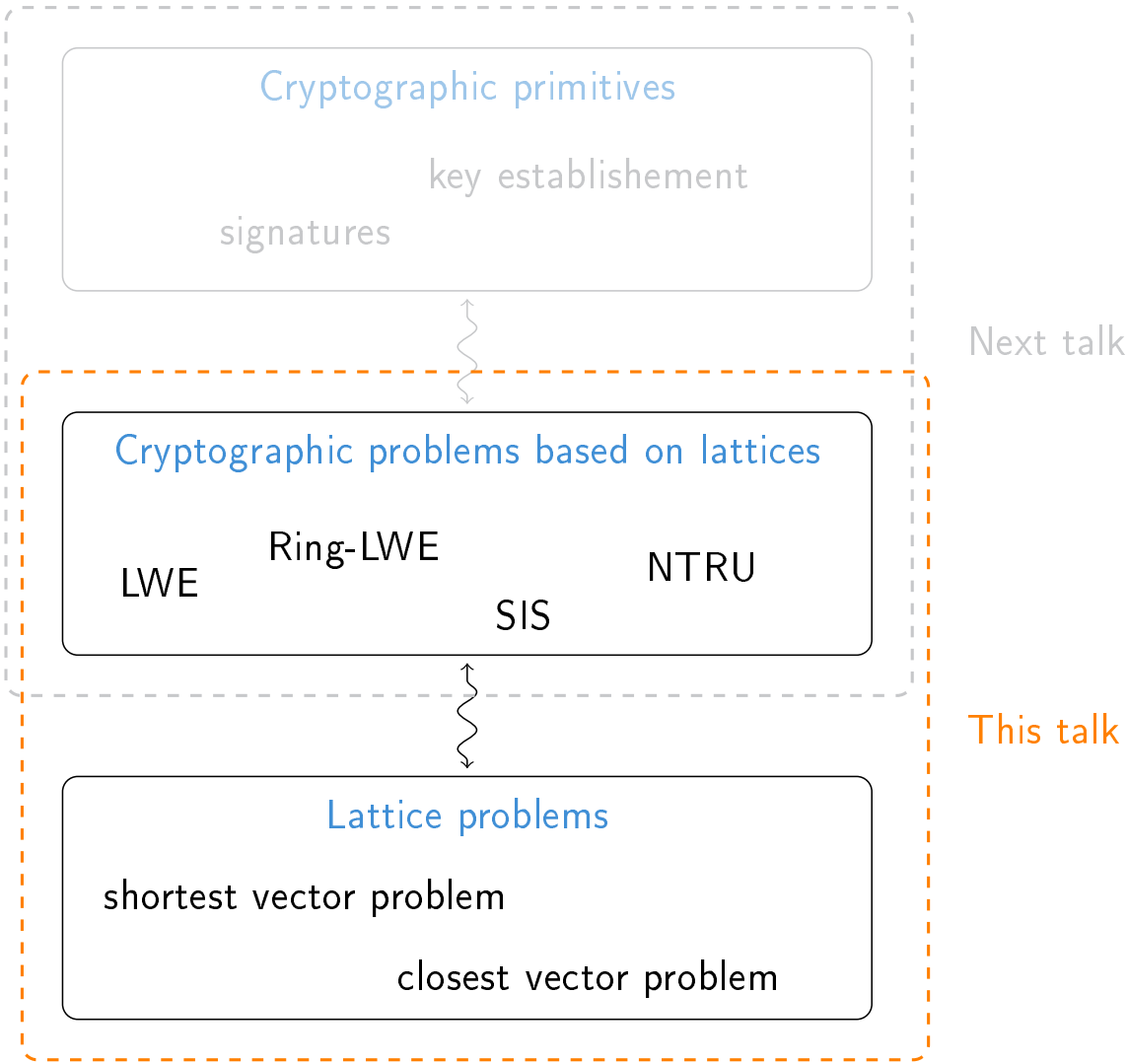
Conclusion

What we have seen



Takeaway: all these problems are supposed to be post-quantum

(for a good choice of parameters)

After the break

Lattice-based crypto, part 2:

How to construct cryptographic primitives from all these problems?