Overview of attacks on ideal lattices

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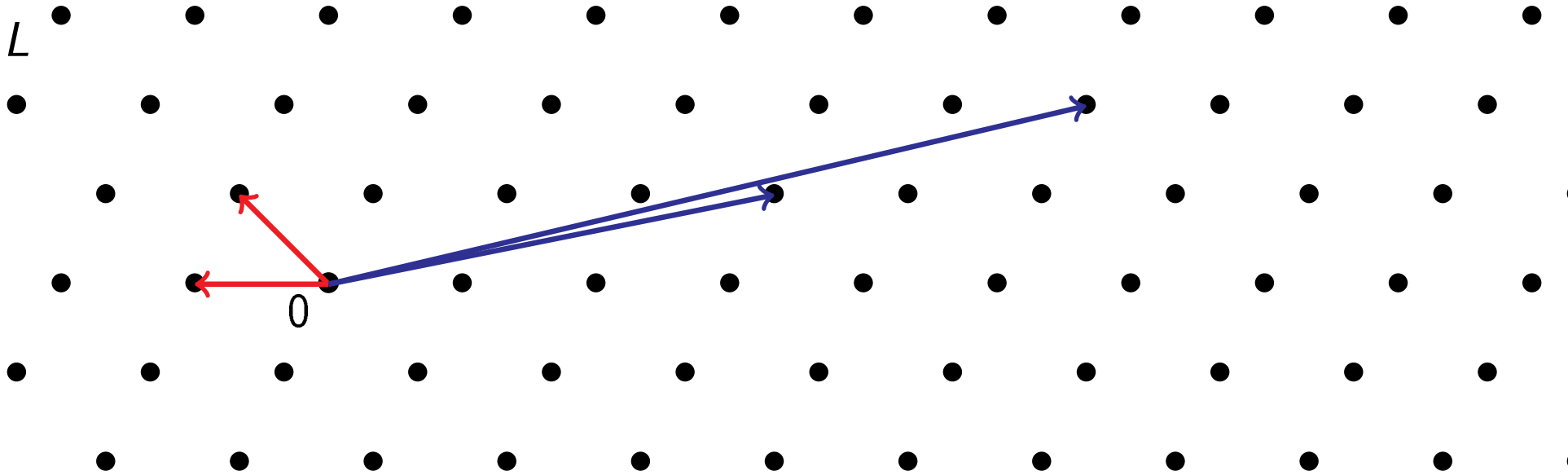
CNRS and Université de Bordeaux

talk at ANSSI

Outline of the talk

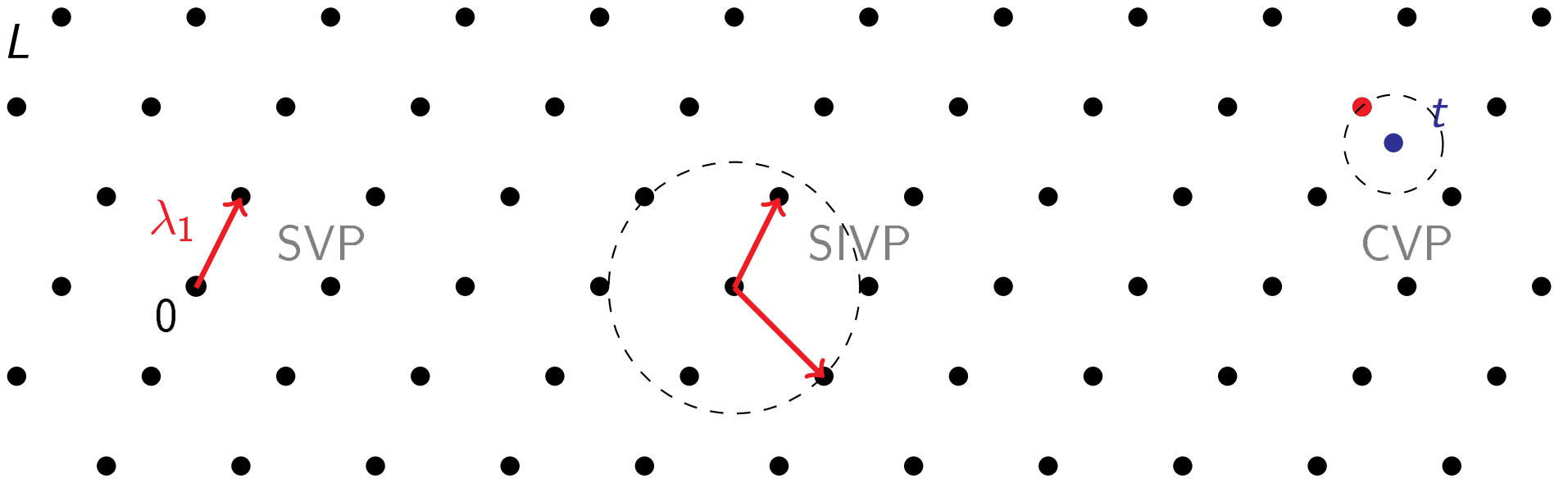
First definitions

Lattices



- $L = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
- $B \in \mathrm{GL}_n(\mathbb{R})$ is a basis
- n is the dimension of L

Algorithmic problems



SVP
shortest vector problem

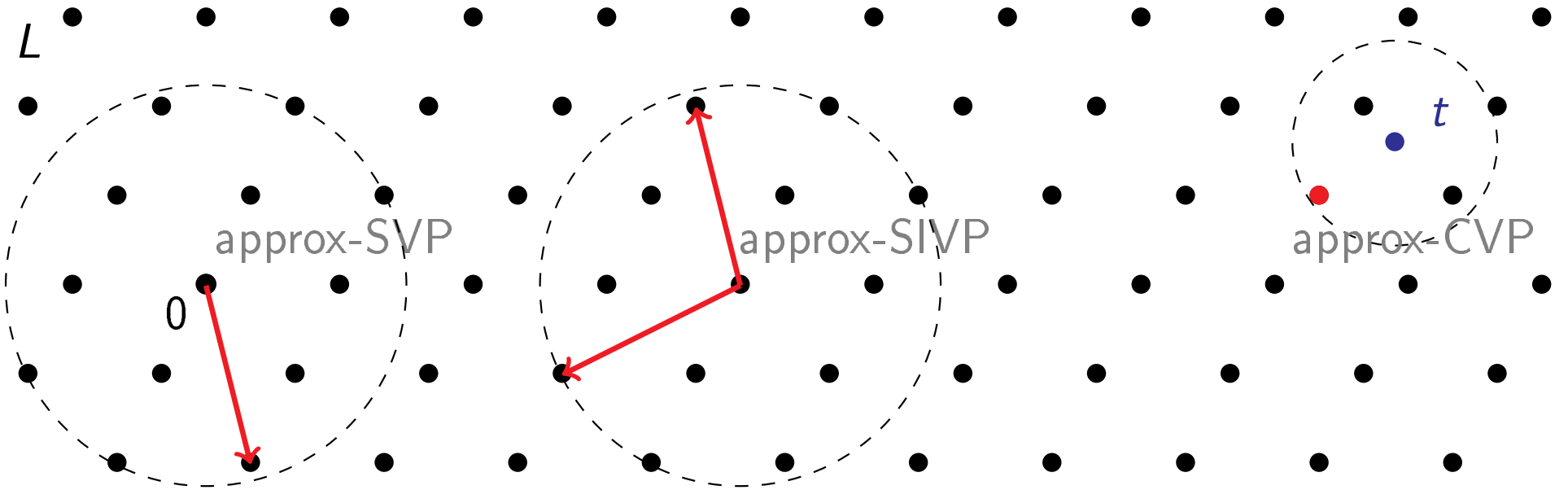
SIVP
shortest independent vector problem

CVP
closest vector problem

Supposedly hard to solve when n is large (input: a bad basis of L)

- even with a quantum computer
- even with a small approximation factor $(\mathrm{poly}(n))$

Algorithmic problems



approx-SVP
shortest vector problem

approx-SIVP
shortest independent vector problem

approx-CVP
closest vector problem

Supposedly hard to solve when n is large (input: a bad basis of L)

- even with a quantum computer
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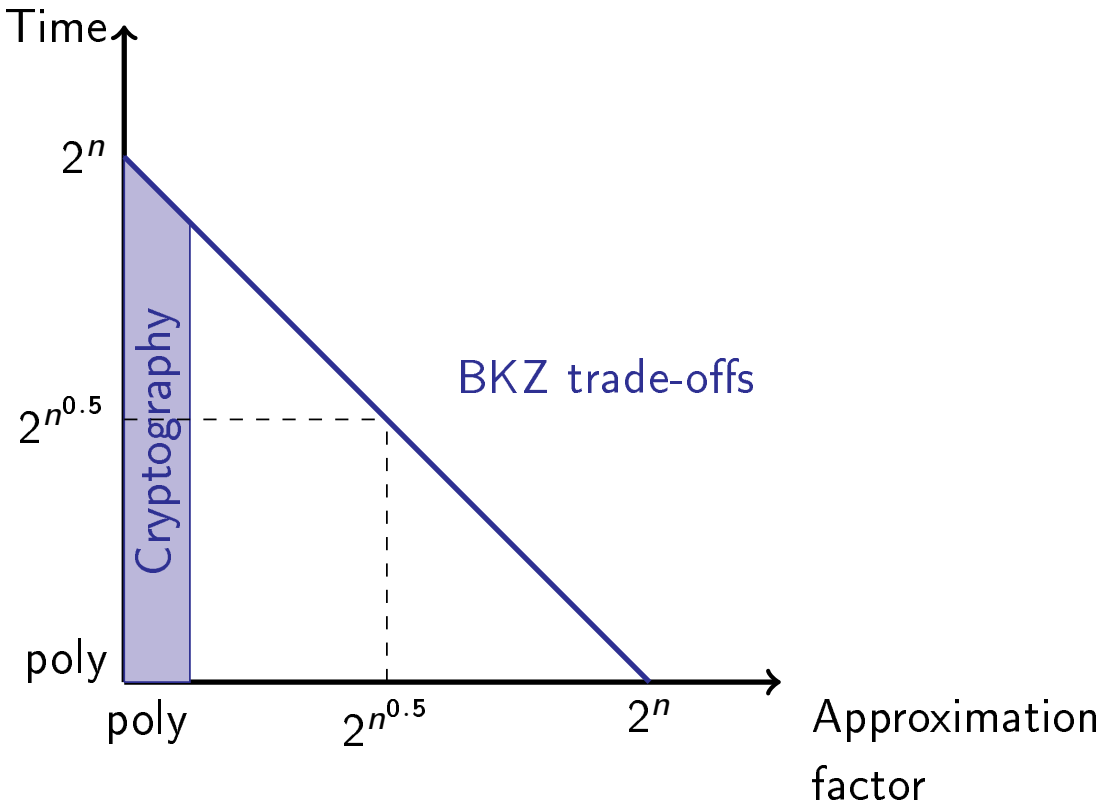
Hardness of SVP and CVP

Best Time/Approximation trade-off for SVP, CVP (even quantumly):
BKZ algorithm [Sch87,SE94]



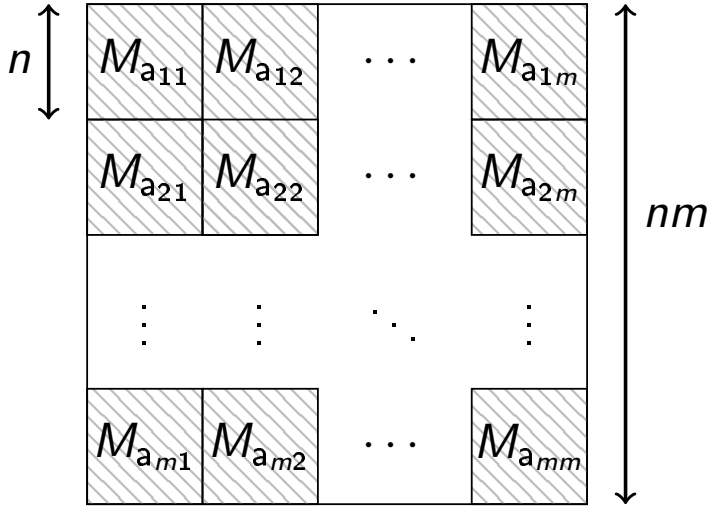
[Sch87] C.-P. Schnorr. A hierarchy of polynomial time lattice basis reduction algorithms. TCS.

[SE94] C.-P. Schnorr and M. Euchner. Lattice basis reduction: improved practical algorithms and solving subset sum problems. Mathematical programming.

Structured lattices

$$M_{\mathsf{a}} = \begin{pmatrix} a_1 & -a_n & \cdots & -a_2 \\ a_2 & a_1 & \cdots & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_1 \end{pmatrix}$$

basis of a special case of
ideal lattice



basis of a special case of
module lattice
of rank m

ideal-SVP = SVP restricted to ideal lattices
module-SVP = SVP restricted to module lattices
$\Rightarrow$ hardness of these restricted problems much less understood than SVP

Context

Standard lattice-based problems

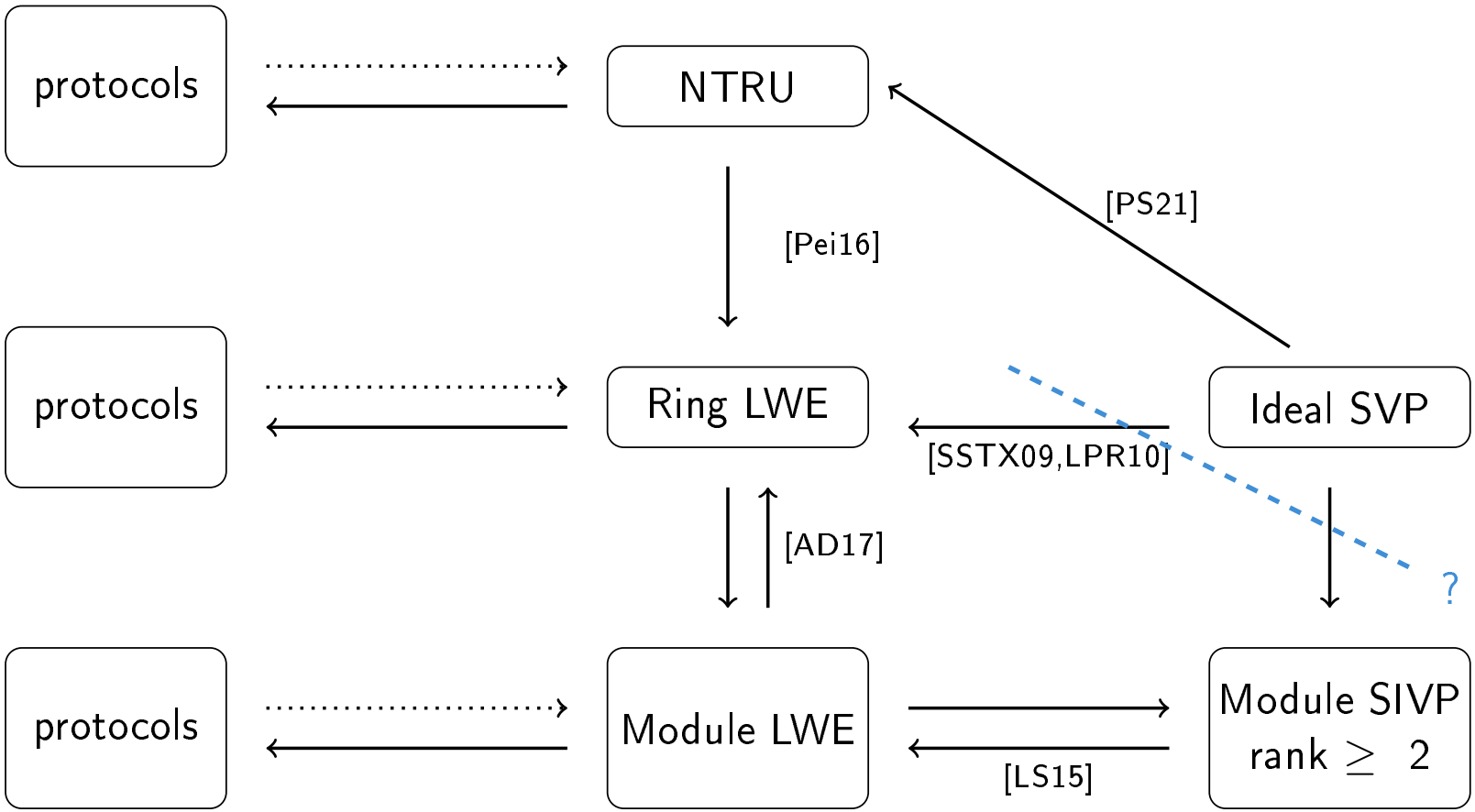
LWE

- post-quantum
- equivalent to worst-case SIVP in unstructured lattice
- not super efficient

RLWE / Module-LWE / NTRU

- post-quantum
- efficient
- how do they compare to structured lattice problems?

Reductions



⚠ Arrows may not all compose (different parameters) ⚠

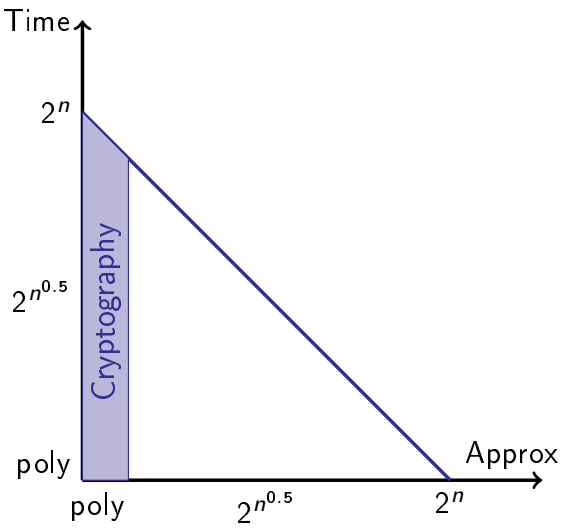
Summing up

breaking ideal-SVP $\not\Rightarrow$ breaking RLWE / module-LWE / NTRU

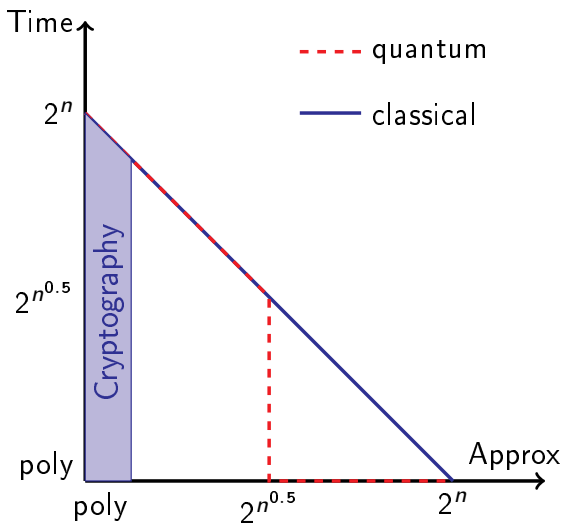
- as long as the attack does not generalize to rank ≥ 2, we are safe
- belief that there is a gap between rank 1 (ideals) and rank ≥ 2

State-of-the-art for ideal-SVP

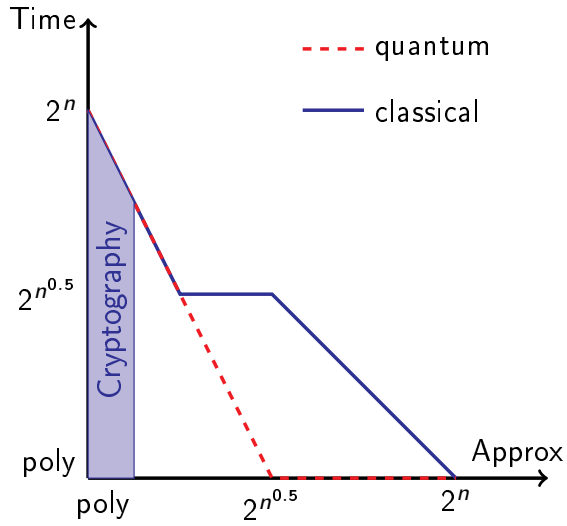
How easy is ideal-SVP compared to SVP?



Unstructured lattices



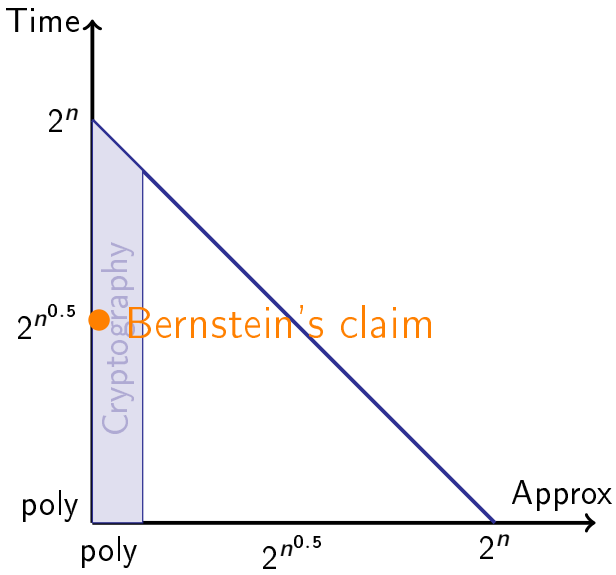
ideals lattices [CDW17]
(in cyclotomic fields)



ideals lattices [PHS19,BR20]
(with $2^{O(n)}$ pre-processing)

- almost no impact for crypto size params
- no reduction from RLWE to ideal-SVP

Bernstein's claim



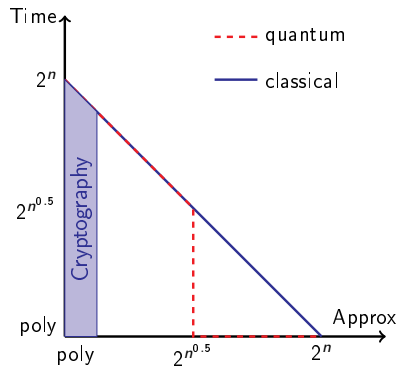
Ideal lattices

(power-of-two cyclotomic)

- This is in the crypto regime

  (but still ideal-SVP $\not\to$ RLWE)

⚠ So far, no argument to support this claim



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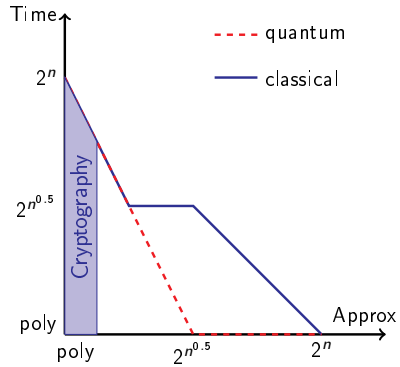


ideals lattices [PHS19,BR20]

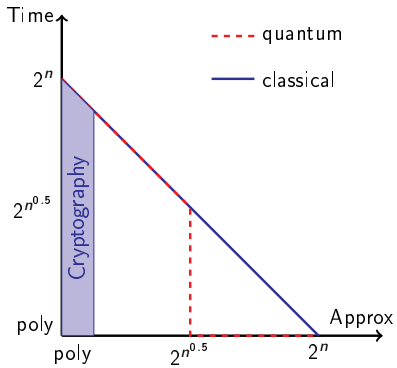
(with $2^{O(n)}$ pre-processing)

Concrete impact

When does [CDW17] starts out-performing BKZ?

- For reasonable run-time (a few core-days):
  - dim $\gtrsim 5\,000$
- For NIST's weakest security requirement:
  - dim $\gtrsim 17\,000$

- Dimension of NIST candidates:
  - dim ≈ 500 or $1\,000$



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Techniques

Math background

Notation

$K = \mathbb{Q}[X]/(X^n + 1)$, with $n = 2^k$ (or any cyclotomic field)

$O_K = \mathbb{Z}[X]/(X^n + 1)$

- Units: $O_K^\times = \{a \in O_K \mid \exists b \in O_K, ab = 1\}$
- Principal ideals: $\langle g \rangle = \{gr \mid r \in O_K\}$
  - g is a generator of $\langle g \rangle$
  - $\{$ generators of $\langle g \rangle$ $\} = \{gu \mid u \in O_K^\times\}$

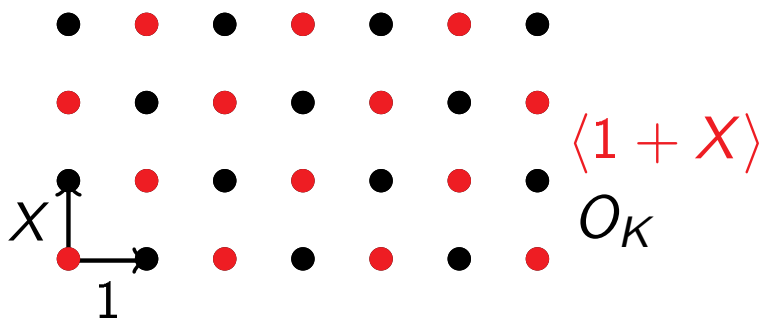
Why is $\langle g \rangle$ a lattice?

O_K is a lattice

$$O_K = \mathbb{Z}[X]/(X^n + 1) \;\to\; \mathbb{R}^n$$

$$r(X) = \sum_{i=0}^{n-1} r_i X^i \;\mapsto\; (r_0, r_1, \ldots, r_{n-1})$$

$$\begin{cases} \langle g \rangle \subseteq O_K \simeq \mathbb{Z}^n \\ \text{stable by '+' and '}-\text{'} \end{cases} \Rightarrow \text{ideal lattice}$$



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Basis: $g, gX, gX^2, \ldots, gX^{n-1}$

$$\text{i.e.,} \quad \begin{pmatrix} g_0 & -g_{n-1} & \cdots & -g_1 \\ g_1 & g_0 & \cdots & -g_2 \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-1} & g_{n-2} & \cdots & g_0 \end{pmatrix}$$

The Log space

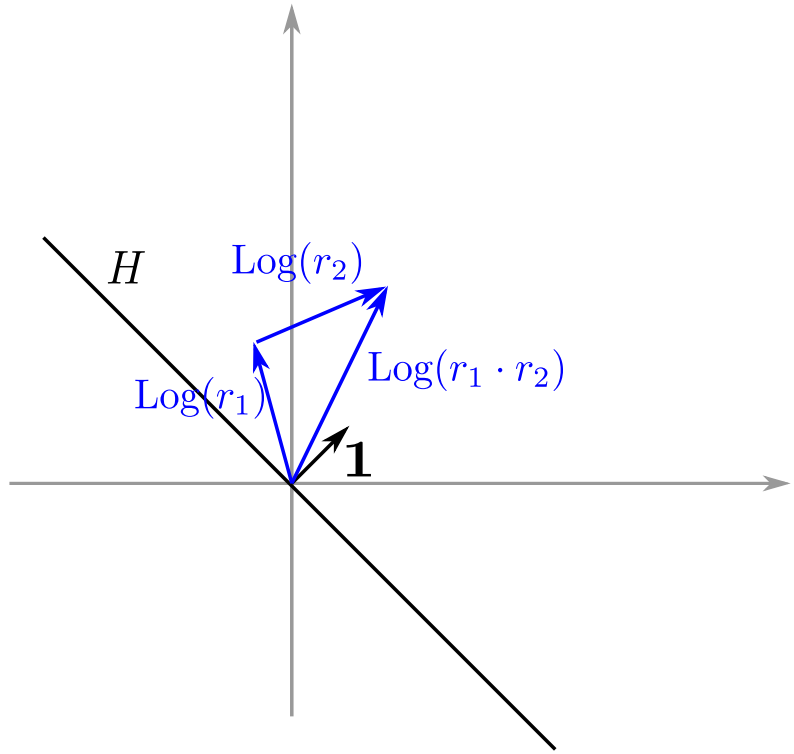
$\mathrm{Log} : O_K \to \mathbb{R}^n$ (take the log of every coordinate)

Let $1 = (1, \cdots, 1)$ and $H = 1^{\perp}$.

Properties ($r \in O_K$)

$\mathrm{Log}\, r = h + a \cdot 1$, with $h \in H$

- $\mathrm{Log}(r_1 \cdot r_2) = \mathrm{Log}(r_1) + \mathrm{Log}(r_2)$



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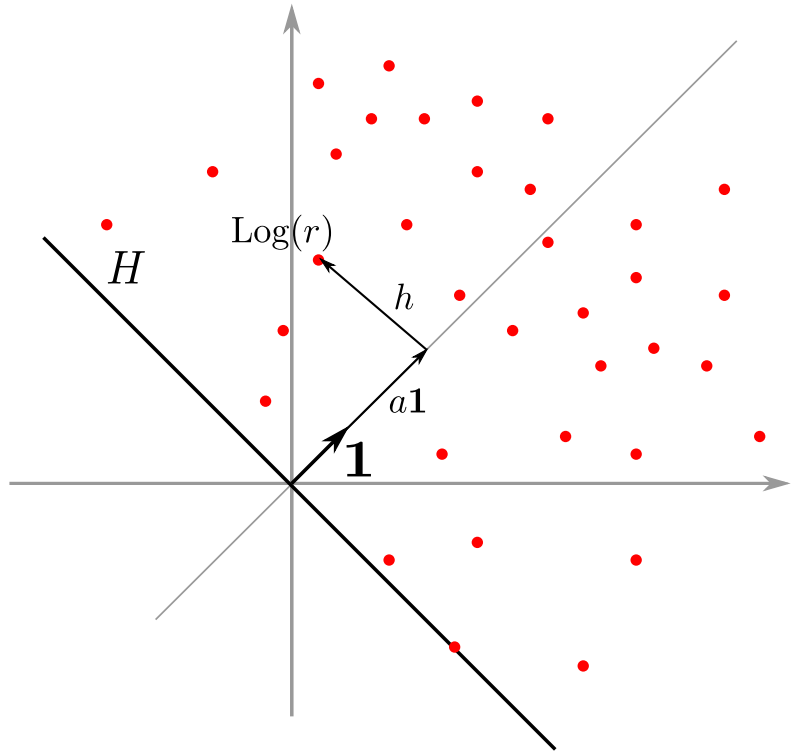
- $a \geq 0$



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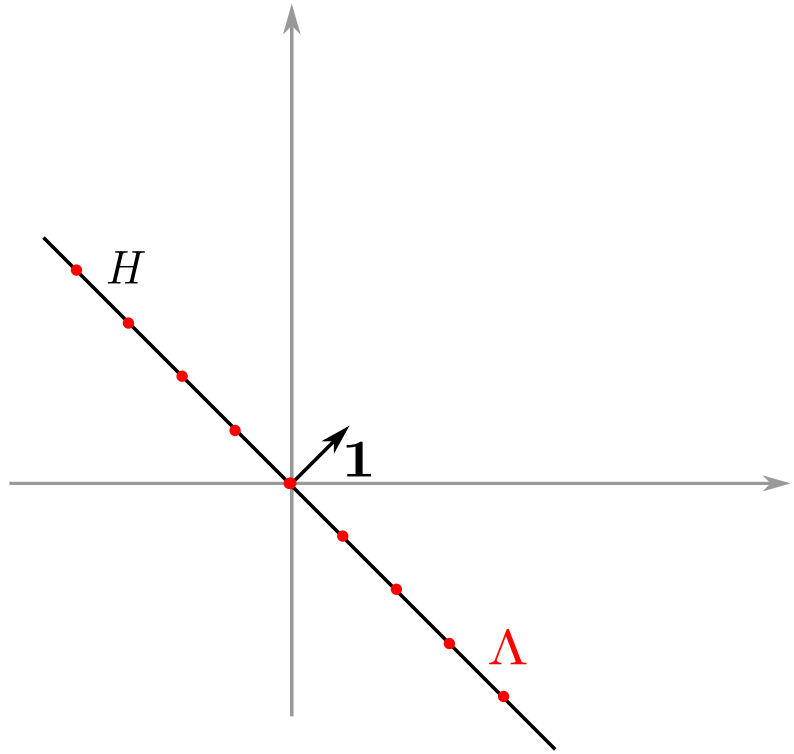
- $\mathrm{Log}(r_1 \cdot r_2) = \mathrm{Log}(r_1) + \mathrm{Log}(r_2)$
- $a \geq 0$
- $a = 0$ iff r is a unit
- $\|r\| \simeq \exp(\|\mathrm{Log}\, r\|_\infty)$

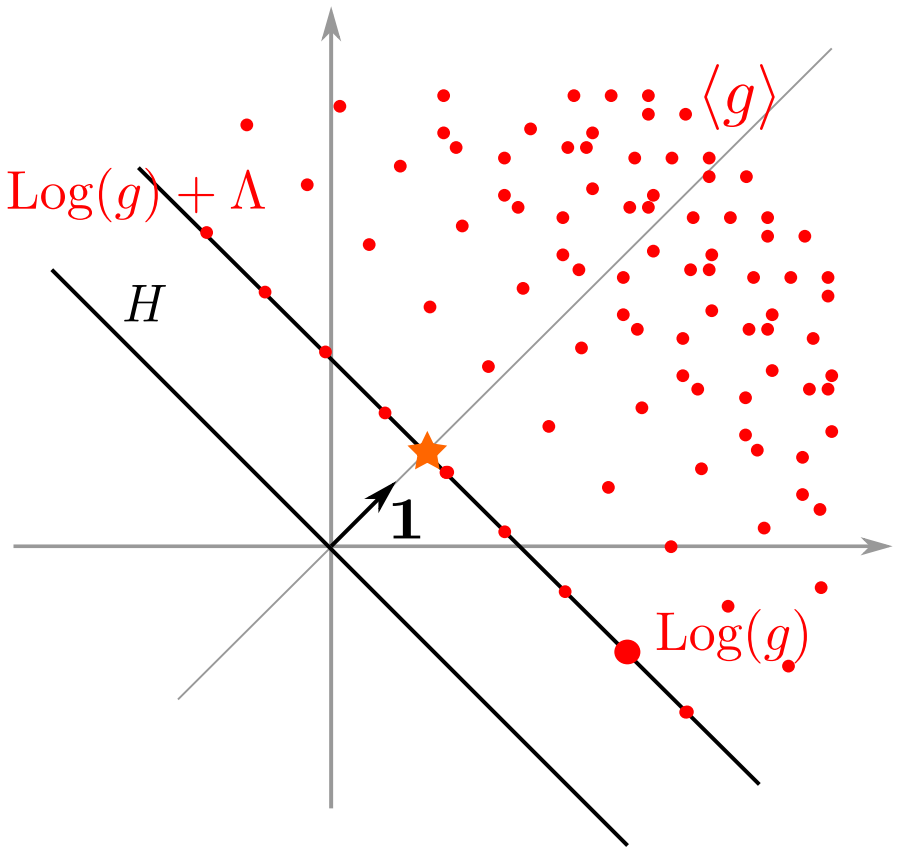


The Log unit lattice

$\Lambda := \mathrm{Log}(O_K^\times)$ is a lattice in H.

The basic algorithm [CGS14,CDPR16]

What does $\mathrm{Log}\langle g \rangle$ look like?

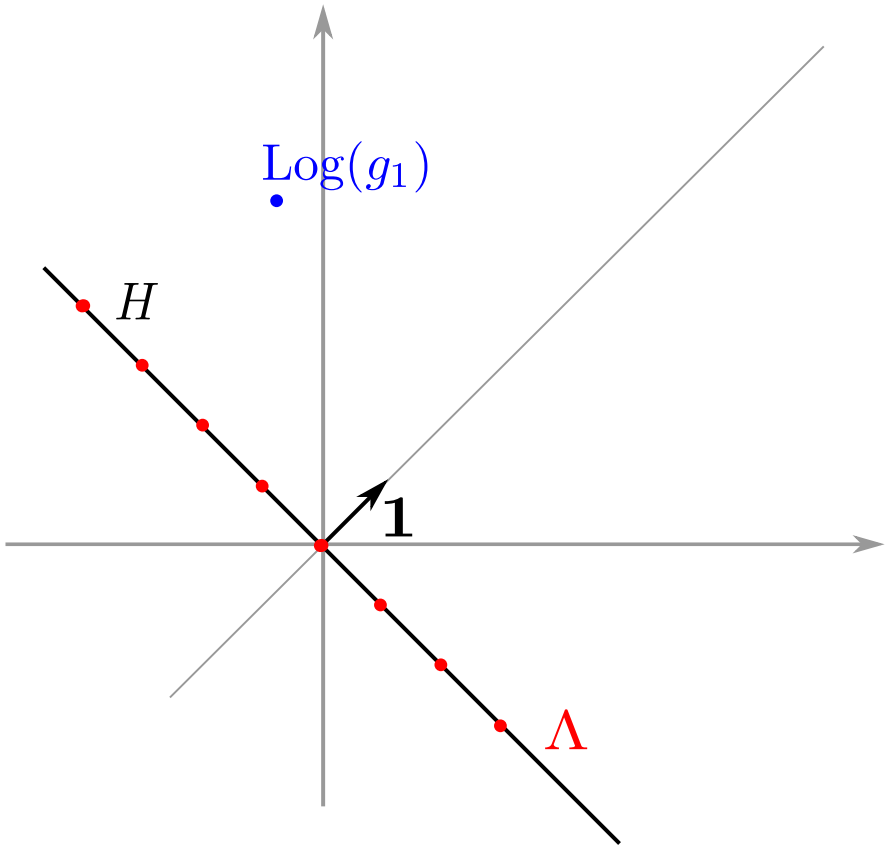


[CGS14]: Campbell, Groves, and Shepherd. Soliloquy: a cautionary tale.

[CDPR16] Cramer, Ducas, Peikert and Regev. Recovering short generators of principal ideals in cyclotomic rings. EC.

The basic algorithm [CGS14,CDPR16]

- Find a generator g_1 of $\langle g \rangle$.
  - [BS16]: quantum poly time



[BS16]: Biasse, Song. Efficient quantum algorithms for computing class groups and solving the principal ideal problem in arbitrary degree number fields. SODA.

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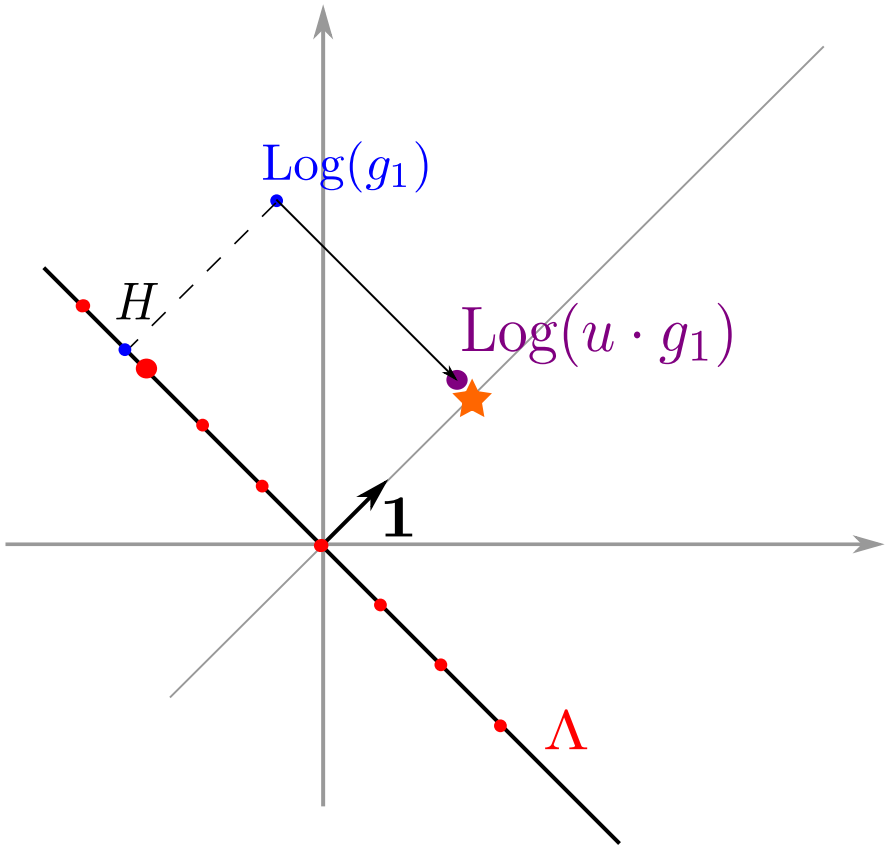

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The basic algorithm [CGS14,CDPR16]

- Find a generator g_1 of $\langle g \rangle$.
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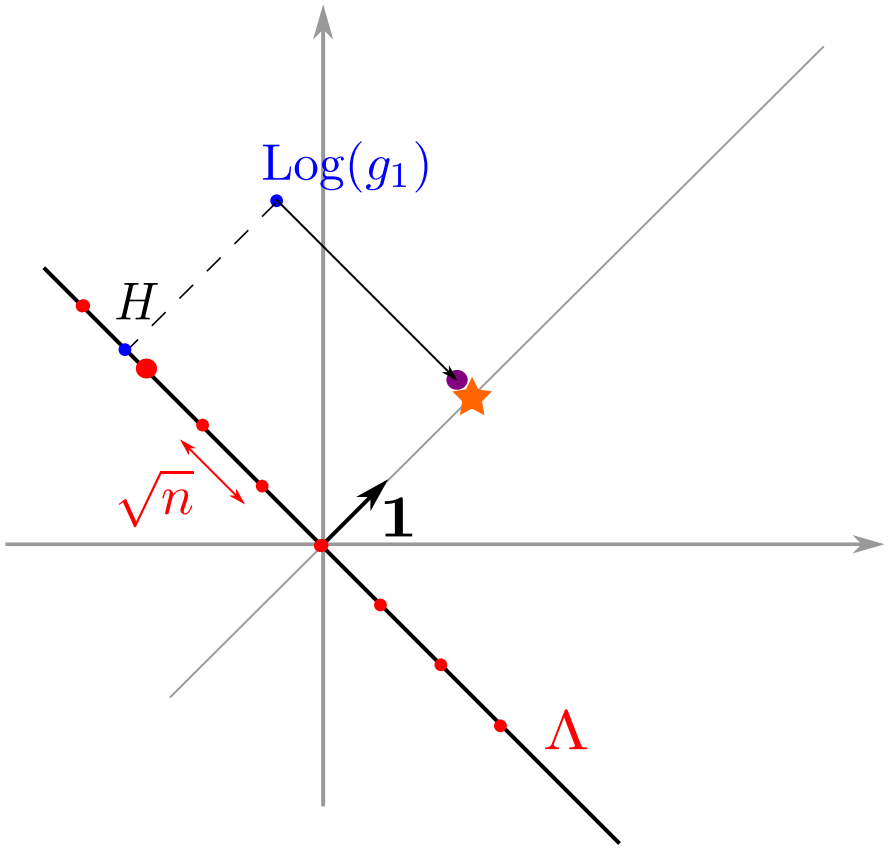
- Solve CVP in Λ
  - Good basis of Λ (cyclotomic field)

    $\Rightarrow$ CVP in poly time

    $\Rightarrow \|h\| \leq \widetilde{O}(\sqrt{n})$

$$\|ug_1\| \leq 2^{\widetilde{O}(\sqrt{n})} \cdot \lambda_1$$



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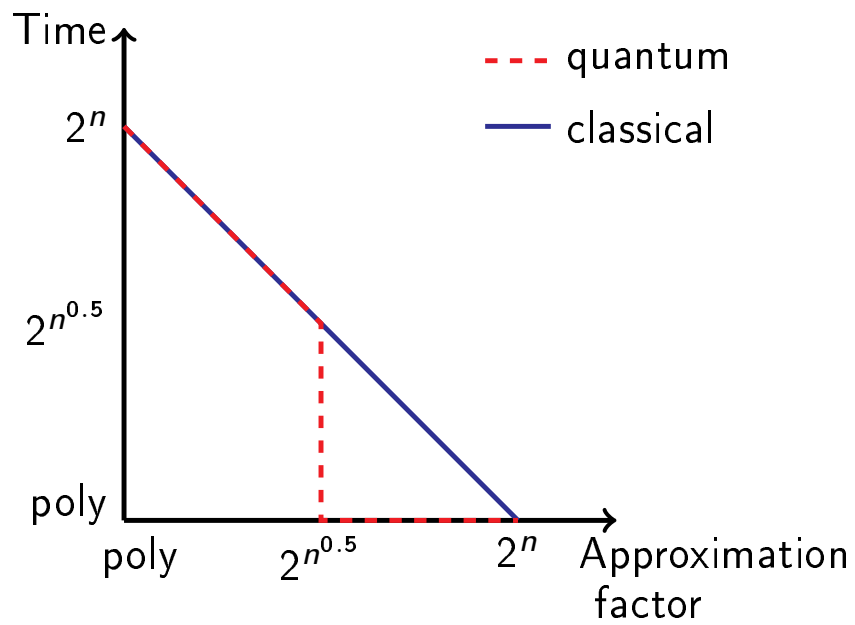


- Heuristic
- Cyclotomic fields

[BS16]: Biasse, Song. Efficient quantum algorithms for computing class groups and solving the principal ideal problem in arbitrary degree number fields. SODA.

More evolved algorithms: using S-units

Idea: replace units by S-units

+ covering radius of Log-S-unit lattice $= O(1)$ (instead of $O(\sqrt{n})$)
  - can reach approximation factor $\mathrm{poly}(n)$ (instead of $2^{O(\sqrt{n})}$)

− we don't know a good basis of the Log-S-unit lattice
  - need to pre-compute it (time $2^{O(n)}$)
  - even with the best basis possible,
    we can only solve CVP with approx $O(\sqrt{n})$ in poly time
    $\Rightarrow$ still $2^{O(\sqrt{n})}$ approx-SVP in poly time

Conclusion

Attacks on module lattices

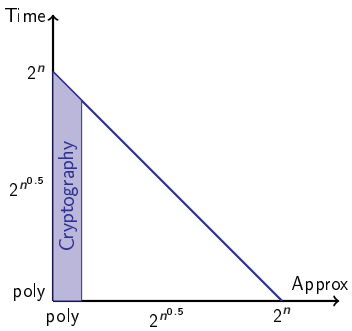
Not much:

- BKZ algorithm for modules [MS20]
  - does not outperform BKZ, but the algo uses only modules (no unstructured lattices)
- Algorithm for SVP in rank-2 modules [LPSW19]
  - Needs an oracle solving CVP in a fixed lattice of dimension n^2

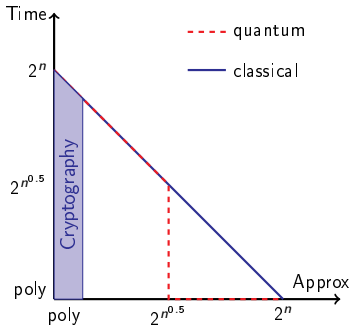
[MS20] Mukherjee and Stephens-Davidowitz. Lattice reduction for modules, or how to reduce module-SVP to module-SVP. Crypto.

[LPSW19] Lee, Pellet-Mary, Stehlé, Wallet. An LLL algorithm for module lattices. Asiacrypt.

Conclusion



Unstructured lattices
and module lattices



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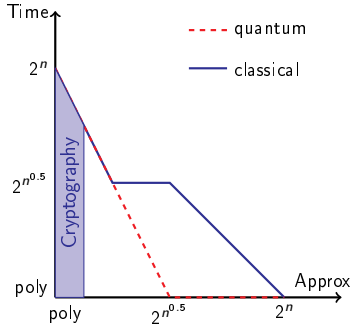

ideals lattices [PHS19,BR20]
(with $2^{O(n)}$ pre-processing)

Missing ingredients to have an impact on crypto:

- Attacks on modules of rank ≥ 2 (not ideals)
- Attacks for small approximation factors

To learn more: Damien Stehlé's invited talk at PQCrypto 2021[1]

Thank you

[1] https://pqcrypto2021.kr/download/program/3.1_PQC.pdf

References

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[SSTX09] Lyubashevsky, Peikert, Regev. On ideal lattices and learning with errors over rings. Eurocrypt.

[LS15] Langlois, Stehlé. Worst-case to average-case reductions for module lattices. DCC.

[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

[AD17] Albrecht, Deo. Large modulus ring-LWE $\geq$ module-LWE. Asiacrypt.

[PS21] Pellet-Mary, Stehlé. On the hardness of the NTRU problem. Asiacrypt.

[CDW17] Cramer, Ducas, Wesolowski. Short stickelberger class relations and application to ideal-SVP. Eurocrypt.

[PHS19] Pellet-Mary, Hanrot, Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

[BR20] Bernard, Roux-Langlois. Twisted-PHS: using the product formula to solve approx-SVP in ideal lattices. AC.