Lattice-based crypto, part 2:
Protocols and structured lattices

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CNRS and university of Bordeaux, France

Summer school in post-quantum cryptography 2022

1-5 August 2022, Budapest

Plan of the talks

Cryptographic primitives

public key encryption

signatures

Cryptographic problems based on lattices

Ring-LWE

LWE

NTRU

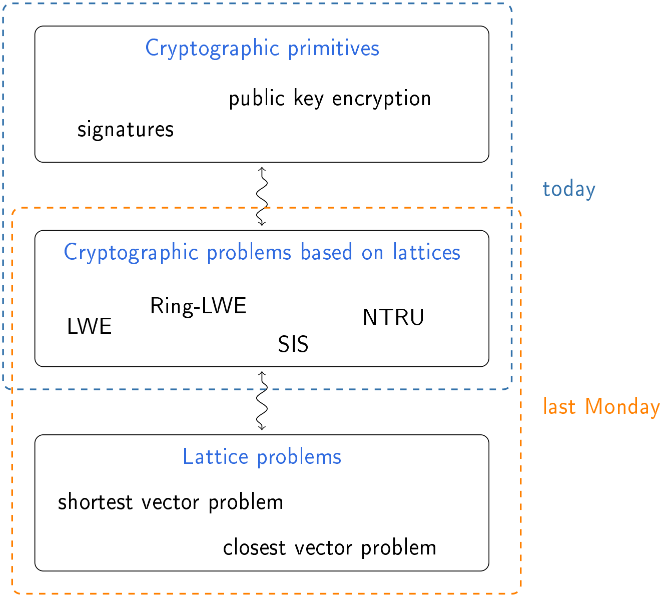
SIS

Lattice problems

shortest vector problem

closest vector problem

Plan of the talks



Outline of the talk

1. Public Key encryption from LWE
2. Trapdoors and signatures
3. Structured lattices

Reminder

What we have seen: LWE and SIS problems

- average case problems
- expressed using simple linear algebra
- best known algorithm takes time $2^{\Omega(n)}$ (if well chosen parameters)
  - even quantumly
  - e.g., $q = \mathrm{poly}(n)$ and $B = \Theta(n)$
- practical hardness quite well understood

Disclaimer

Monday we have seen:

- that LWE/SIS is as hard as worst-case lattice problems
  (i.e., if we can solve LWE/SIS with good proba, we can solve some lattice problem over all lattices)

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  - e.g., binary noise, small modulus q, ...

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But...

- for practical constructions, we choose parameters for which the reductions to worst-case problem do not hold
  - e.g., binary noise, small modulus q, ...
- reductions are used to show that there is no fundamental flaw in the design
  - taking larger parameters, we can prove that the schemes are as secure as worst case lattice problems

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Decision-LWE

χ_B: distribution over $\{-B, \cdots, B\}$

Reminder: decision-LWE

Sample $A \leftarrow \mathrm{Uniform}(\mathbb{Z}_q^{n \times m})$ and s, $e \leftarrow \chi_B^n \times \chi_B^m$

Given A and b, where

$$b := A s + e \bmod q \quad \text{or} \quad b \leftarrow \mathrm{Uniform}(\mathbb{Z}_q^n)$$

Guess whether b is uniform or not.

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Guess whether $\boxed{b}$ is uniform or not.

- assumed to be hard even with a quantum computer
  (for well chosen parameters)

LWE-based encryption scheme [LP11]

Parameters: $n, q \in \mathbb{Z}_{>0}$ and χ_B distribution over $\{-B, \cdots, B\}$

[LP11] Lindner and Peikert. Better key sizes (and attacks) for LWE-based encryption. CT-RSA.

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Encrypt: message $m \in \{0, 1\}$

sample $\tilde{s}^T, \tilde{e}^T \leftarrow \chi_B^n$ and $e' \leftarrow \chi_B$

return $c = (\tilde{s}^T A + \tilde{e}^T, \tilde{s}^T b + e' + m \cdot \lfloor q/2 \rceil)$

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Decrypt: $c = (c_1{}^T, c_2)$

compute $x = c_1{}^T s - c_2 \bmod q$ ($x \in [0, q]$)

return 1 if x is in $[q/4, 3q/4]$ and 0 otherwise

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Correctness

Theorem

If $q \geq 8 \cdot n \cdot B^2 + 4 \cdot B$, then the scheme is correct.

Correctness: for any message m and any $(pk, sk) \leftarrow KeyGen$, it holds that

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Decryption failures: if χ_B is a Gaussian distribution, the scheme might fail with very small probability (χ_B might output something $\geq B$)

Security: high level

Public information:

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$\boxed{b} = \boxed{A}\boxed{s} + \boxed{e}$

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Trapdoors

Two (related) notions of trapdoors for lattices:

- short basis of $\mathcal{L}$
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Short basis

Idea: construct a lattice $\mathcal{L}$ with a good basis B_0 and a bad basis B_1

- given B_1, CVP is hard
- given B_0, CVP is easy

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One can efficiently create a uniform SIS lattice $\mathcal{L}$ together with a short basis of it.

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One can efficiently create a uniform SIS lattice $\mathcal{L}$ together with a short basis of it.

- CVP in $\mathcal{L}$ is hard if SIS is hard (if $\mathcal{L}$ represented by its HNF)
- the short basis enables to solve CVP efficiently

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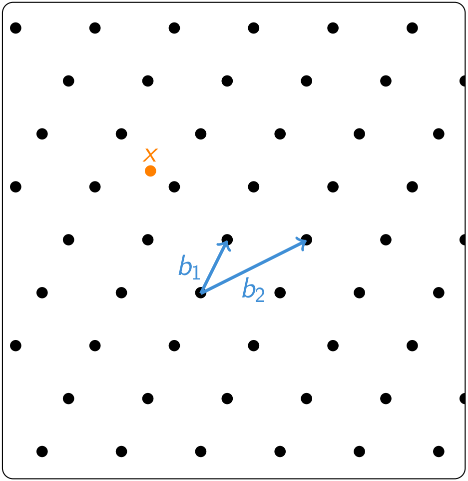
Solving CVP with a short basis



Input: $x = 3.7 \cdot b_1 - 1.4 \cdot b_2$

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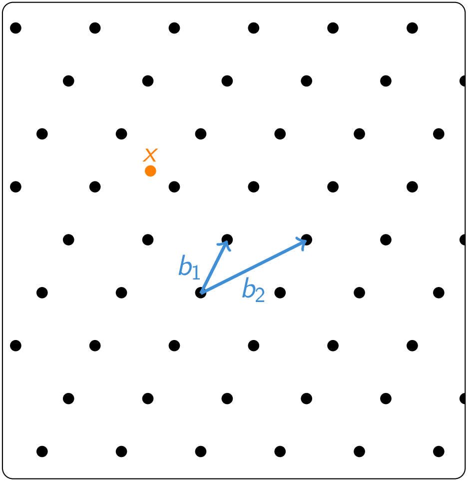

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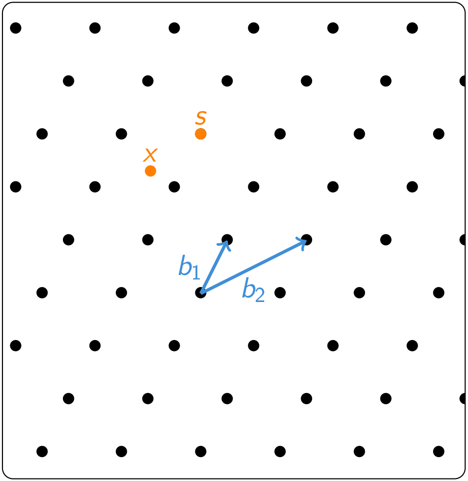

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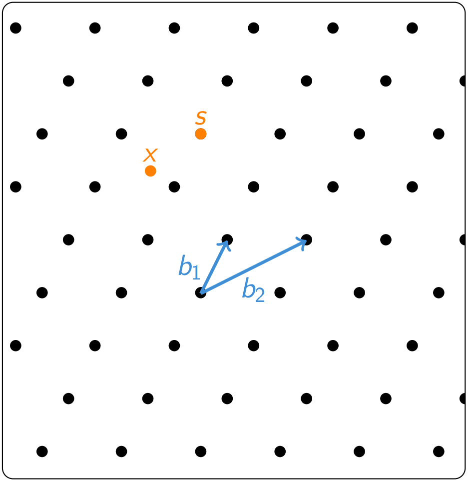

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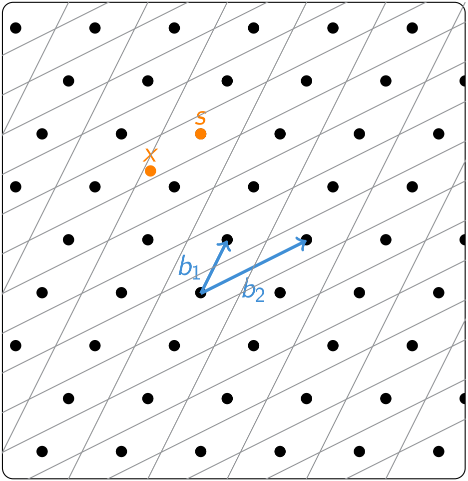

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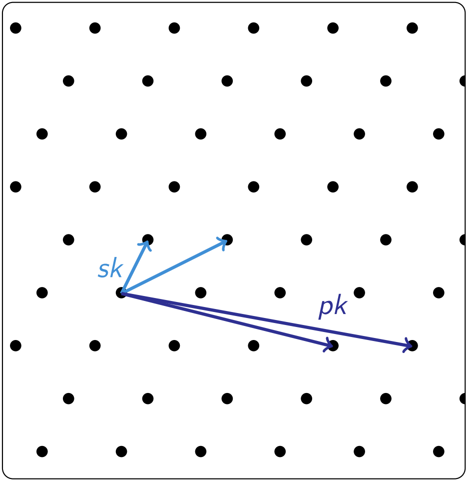
Signing with a trapdoor (hash-and-sign) [GGH97]



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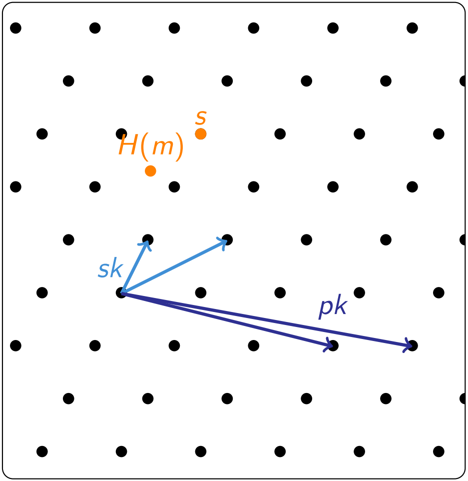

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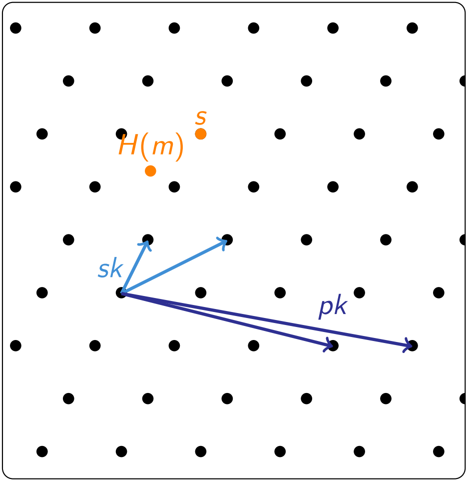

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Verify(s, pk):

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

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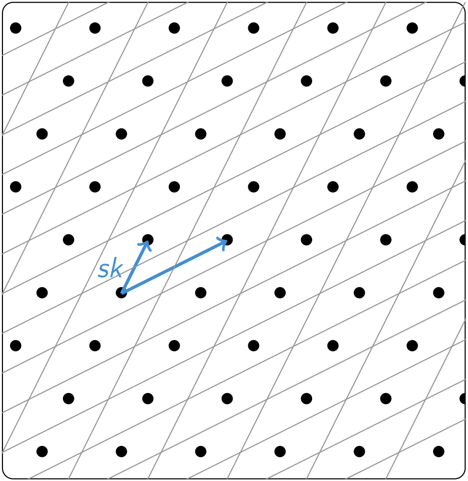
Attack on the signature scheme [NR06]



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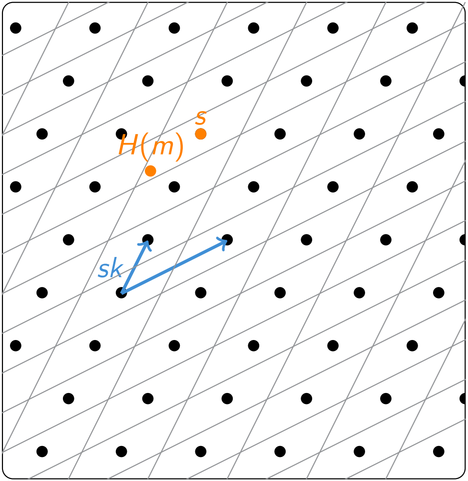

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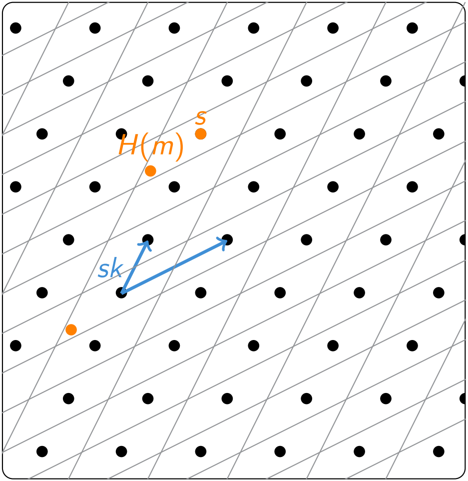

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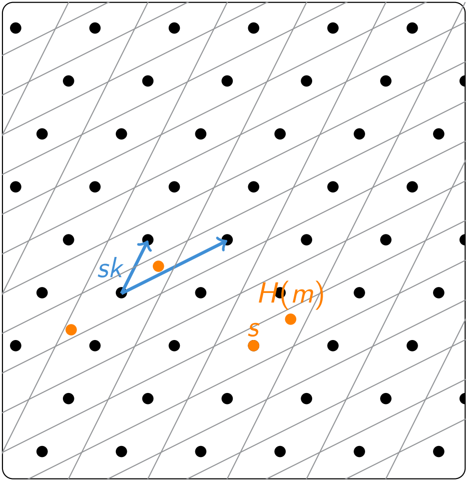

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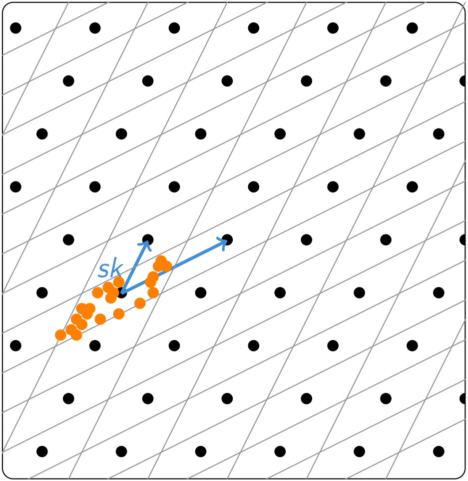

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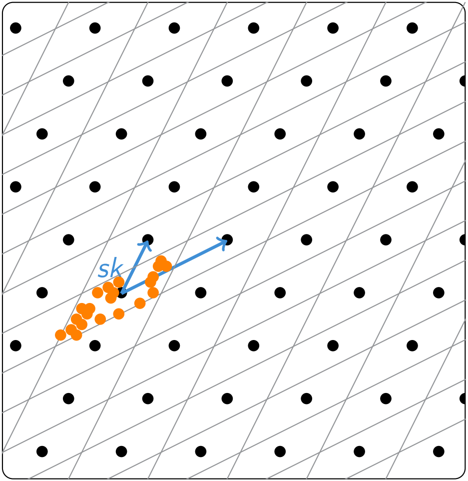

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From the shape of the parallelepiped, one can recover the short basis

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Preventing the attack [GPV08]

Idea: do not solve CVP deterministically but randomly

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Lemma [GPV09]

Assuming that the SIS problem is hard, then the signature scheme is unforgeable under chosen-message attack.

[GPV08] Gentry, Peikert, and Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC

Advanced constructions

One can construct many advanced primitives from lattices:

- (fully) homomorphic encryption
- identity based encryption
- functional encryption for linear functions
- ...

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Ring of integers: $\mathcal{O}_K \subset K$, for this talk $\mathcal{O}_K = \mathbb{Z}[X]/P(X)$
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Embeddings

$(K = \mathbb{Q}[X]/P(X), \quad \alpha_1, \cdots, \alpha_d$ complex roots of $P(X))$

Coefficient embedding:
$$\Sigma : \begin{array}{rcl} K & \to & \mathbb{R}^d \\ \sum_{i=0}^{d-1} y_i X^i & \mapsto & (y_0, \cdots, y_{d-1}) \end{array}$$

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- coefficient embedding is used for constructions (efficient implementation)
- canonical embedding is used in cryptanalysis / reductions (nice mathematical properties)
- for fields used in crypto, both geometries are $\approx$ the same

Ideals

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- $x + y \in I$ for all $x, y \in I$
- $a \cdot x \in I$ for all $a \in \mathcal{O}_K$ and $x \in I$

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- $I_1 = \{2a \,|\, a \in \mathbb{Z}\}$ and $J_1 = \{6a \,|\, a \in \mathbb{Z}\}$ in $\mathcal{O}_K = \mathbb{Z}$
- $I_2 = \{a + b \cdot X \,|\, a + b = 0 \bmod 2,\ a, b \in \mathbb{Z}\}$ in $\mathcal{O}_K = \mathbb{Z}[X]/(X^2 + 1)$

Ideals

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- $I_2 = \{a + b \cdot X \,|\, a + b = 0 \bmod 2,\ a, b \in \mathbb{Z}\} = \langle 1 + X \rangle$

Ideal lattices

$\mathcal{O}_K$ is a lattice:

- $\mathcal{O}_K = 1 \cdot \mathbb{Z} + X \cdot \mathbb{Z} + \cdots + X^{d-1} \cdot \mathbb{Z}$
- $\Sigma(\mathcal{O}_K) = \Sigma(1) \cdot \mathbb{Z} + \cdots + \Sigma(X^{d-1}) \cdot \mathbb{Z}$

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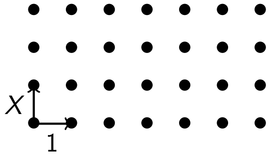
(this is also true for non principal ideals)

(we can replace Σ by σ and $\mathbb{Z}^d$ by $\mathbb{C}^d$)

Ideal lattices (2)



$\Sigma(O_K)$

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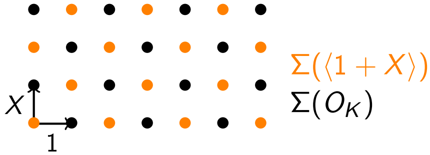

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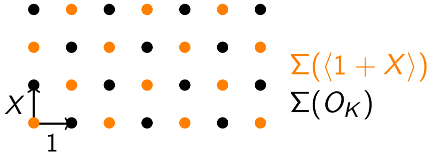

Basis of $\langle g\rangle$: $g, g\cdot X, \cdots, g\cdot X^{d-1}$

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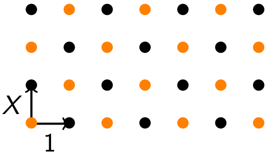

$\Sigma(\langle 1+X \rangle)$

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$$\begin{pmatrix} g_0 & & & \\ g_1 & & & \\ \vdots & & & \\ g_{d-1} & & & \end{pmatrix}$$

(in $K = \mathbb{Q}[X]/X^d + 1$)

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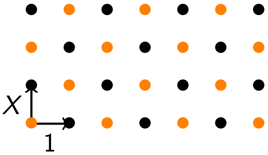

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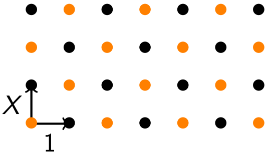

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Module lattices

(Free) module:

$$M = \{B \cdot x \,|\, x \in \mathcal{O}_K^k\} \text{ for some matrix } B \in \mathcal{O}_K^{k \times k} \text{ with } \det_K(B) \neq 0$$

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- k is the module rank
- B is a module basis of M
  (if the module is not free, it has a "pseudo-basis" instead)

$\Sigma(M)$ is a lattice:

- of $\mathbb{Z}$-rank $n := d \cdot k$, included in $\mathbb{Z}^n$

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- of $\mathbb{Z}$-rank $n := d \cdot k$, included in $\mathbb{Z}^n$
- with basis $(\Sigma(b_i X^j))_{\substack{1 \leq i \leq k \\ 0 \leq j < d}}$ (b_i columns of B)

Modules vs ideals

Ideal	=	Module of rank 1
(principal ideal	=	free module of rank 1)

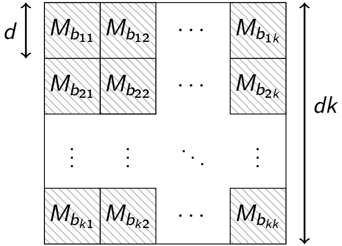
Modules vs ideals

Ideal	$=$	Module of rank 1
(principal ideal	$=$	free module of rank 1)

In $K = \mathbb{Q}[X]/(X^d+1)$:

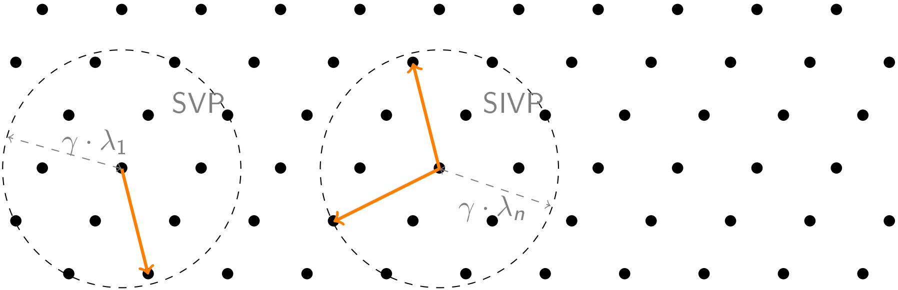
$$M_a = \begin{pmatrix} a_1 & -a_d & \cdots & -a_2 \\ a_2 & a_1 & \cdots & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ a_d & a_{d-1} & \cdots & a_1 \end{pmatrix}$$

basis of a
principal ideal lattice



basis of a free module lattice
of rank k

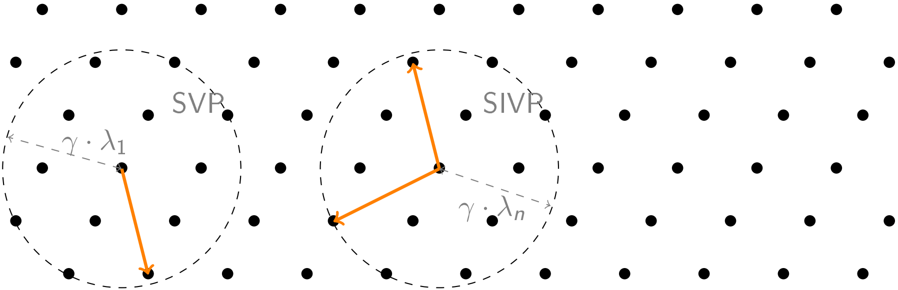
Algorithmic problems



γ-SVP
shortest vector problem

γ-SIVP
shortest independent vector problem

Algorithmic problems



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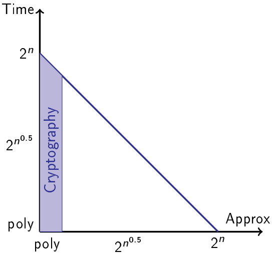
Notations:

- id-X = problem X restricted to ideal lattices
- mod-X_k = problem X restricted to module lattices of rank k

(worst-case: we want algorithms for all ideal/module lattices)

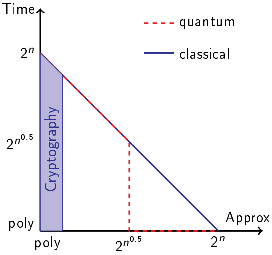
Hardness of SVP

Asymptotics:



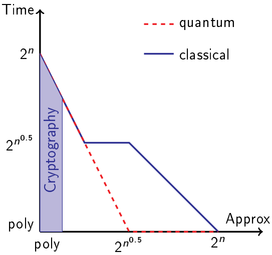
SVP and mod-SVP$_k$

($k \geq 2$)



id-SVP [CDW17]

(in cyclotomic fields)



id-SVP [PHS19,BR20]

(with $2^{O(n)}$ pre-processing)

[CDW17] Cramer, Ducas, Wesolowski. Short stickelberger class relations and application to ideal-SVP. Eurocrypt.

[PHS19] Pellet-Mary, Hanrot, Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

[BR20] Bernard, Roux-Langlois. Twisted-PHS: using the product formula to solve approx-SVP in ideal lattices. AC.

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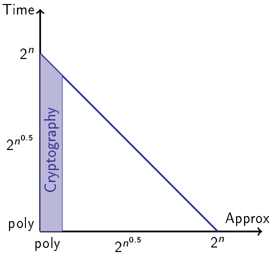

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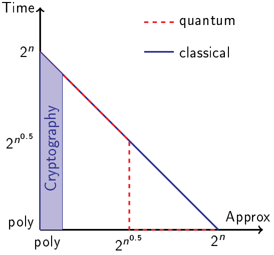

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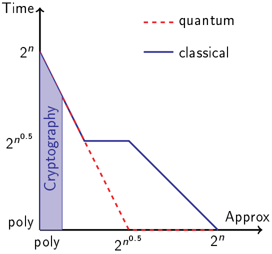

id-SVP [PHS19,BR20]
(with $2^{O(n)}$ pre-processing)

Practice: Darmstadt challenge[1]

$\rightsquigarrow$ max dim for SVP: 180
$\rightsquigarrow$ max dim for id-SVP: 150

[1] https://www.latticechallenge.org/

RLWE and mod-LWE

Ring and Module-LWE

(search) mod-LWE$_k$

Parameters: q and B

Problem: Sample

- $A \leftarrow \text{Uniform}((\mathcal{O}_K/(q\mathcal{O}_K))^{m\times k})$
- $s, e \in \mathcal{O}_K^k \times \mathcal{O}_K^m$ with coefficients in $\{-B, \cdots, B\}$

Given A and $b = A \cdot s + e \bmod q$, recover s

(size of s and e can be defined using Σ or σ)

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$$\boxed{\text{RLWE} = \text{mod-LWE}_1}$$

mod-LWE vs mod-SIVP

$$\text{mod-SVP}_m \quad \geq \quad \text{mod-LWE}_k \quad \geq \quad \text{mod-SIVP}_k$$

quantumly!

mod-LWE vs mod-SIVP

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How large should m be?

- as small as possible
- but so that the closest point to b is As

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- $m = k$ is not sufficient

mod-LWE vs mod-SIVP

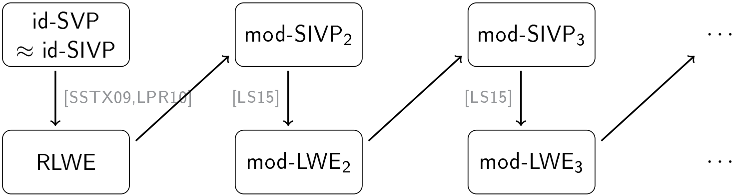
$$\text{mod-SVP}_m \quad \geq \quad \text{mod-LWE}_k \quad \geq \quad \text{mod-SIVP}_k$$

quantumly!

How large should m be?

- as small as possible
- but so that the closest point to b is As
- $m = k$ is not sufficient
- $m = k + 1$ might be sufficient depending on B and q
  - we need roughly $m = k \cdot \frac{\log(q)}{\log(q/B)}$
  - for $k = 1$, $m = 2$ is possible if $B \lesssim \sqrt{q}$

Reductions



⚠ Arrows may not all compose (different parameters) ⚠

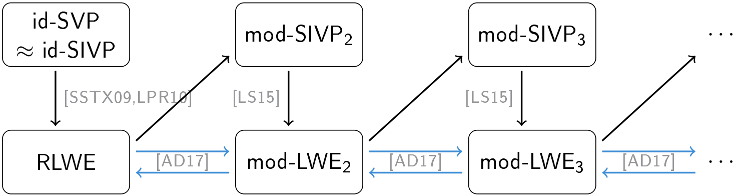
- References are for the first reductions. Better, more recent reductions may exist.
- reductions may be quantum
- reductions hold for σ and Gaussian noise

[SSTX09] Stehlé, Steinfeld, Tanaka, Xagawa. Efficient public key encryption based on ideal lattices. Asiacrypt.

[LPR10] Lyubashevsky, Peikert, Regev. On ideal lattices and learning with errors over rings. Eurocrypt.

[LS15] Langlois, Stehlé. Worst-case to average-case reductions for module lattices. DCC.

Reductions



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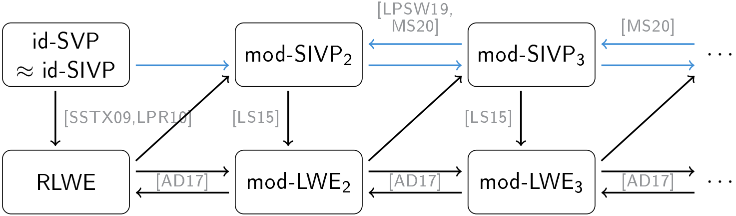
[AD17] Albrecht, Deo. Large modulus ring-LWE $\geq$ module-LWE. Asiacrypt.

Reductions



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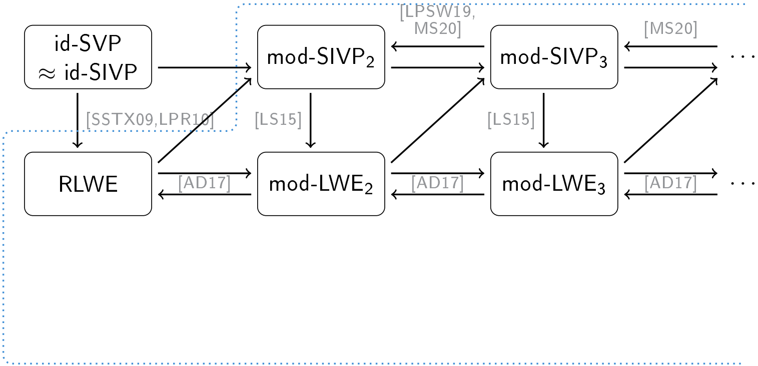

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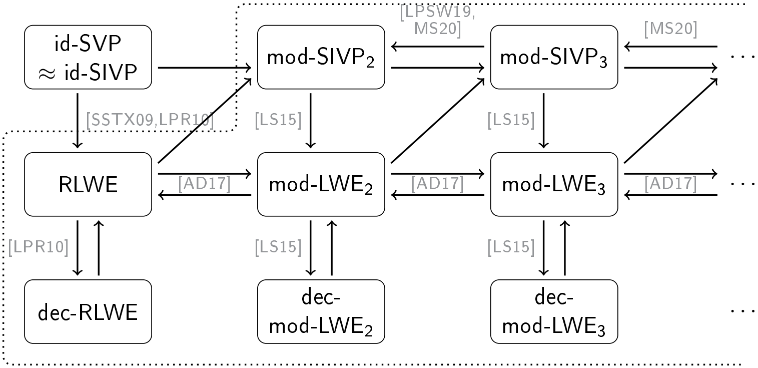

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NTRU

NTRU [HPS98]

(search) NTRU

Parameters: $q \geq B > 1$

Objective: Sample $f, g \in \mathcal{O}_K$ with coefficients in $\{-B, \cdots, B\}$.
Given $h = f \cdot g^{-1}$, recover (f, g)

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

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dec-NTRU

Parameters: q, B

Objective: distinguish between h as above and h uniform in $\mathcal{O}_K/(q\mathcal{O}_K)$

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Exercise: why is it unsafe to take $h = f$ or $h = g^{-1} \bmod q$?

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Objective: distinguish between h as above and h uniform in $\mathcal{O}_K/(q\mathcal{O}_K)$

Exercise: why is it unsafe to take $h = f$ or $h = g^{-1} \bmod q$?

- f is small, easy to distinguish from Uniform$(\mathcal{O}_K/(q\mathcal{O}_K))$ (which is likely $\approx q$)
- if $h = g^{-1}$, one can compute $h^{-1} = g \bmod q$ and same situation as above

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

Two regimes of NTRU

If $B \geq \sqrt{q} \cdot \mathrm{poly}(d)$

If $B \leq \sqrt{q}/\mathrm{poly}(d)$

Two regimes of NTRU

If $B \geq \sqrt{q} \cdot \mathrm{poly}(d)$

- h is statistically close to uniform mod q [SS11,WW18]
- dec-NTRU is statistically hard

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- h is not statistically close to uniform mod q
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If $B \leq \sqrt{q}/\mathrm{poly}(d)$

- h is not statistically close to uniform mod q
- NTRU is a special case of unique-SVP

For the rest of the talk, we consider $B \ll \sqrt{q}$

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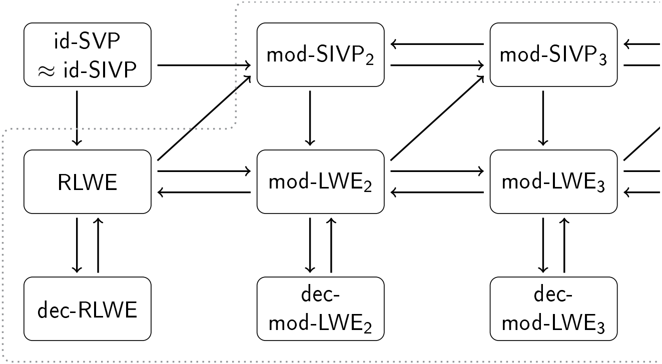

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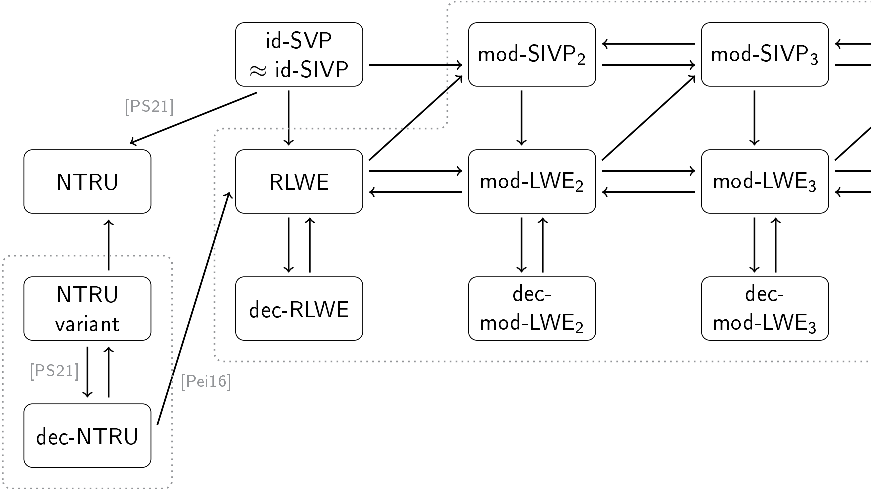

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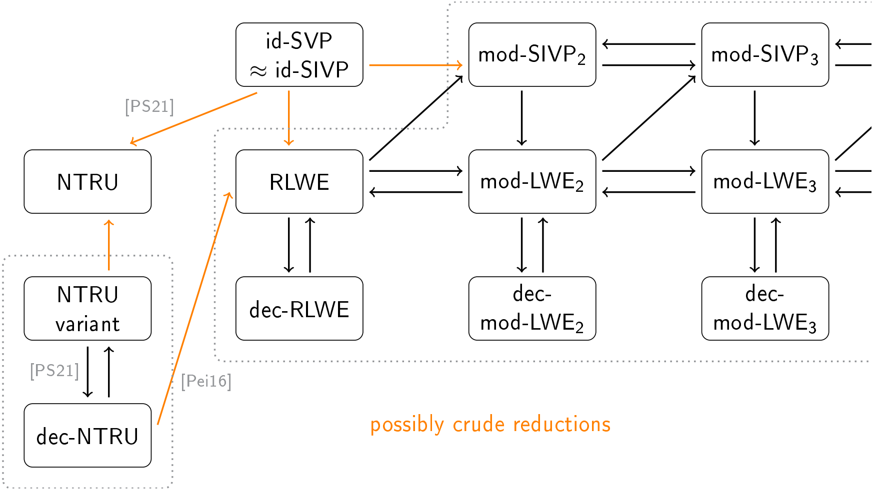
[PS21] Pellet-Mary, Stehlé. On the hardness of the NTRU problem. Asiacrypt.

Reductions



⚠ Arrows may not all compose (different parameters) ⚠

[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

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Take-away from this section

id-SVP is a lower bound
on the hardness of RLWE, mod-LWE, NTRU

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Breaking id-SVP do break:

- some early FHE schemes
- the PV-Knap problem [HPS+14,BSS22]

[HPS+14] Hoffstein, Pipher, Schanck, Silverman, and Whyte. Practical signatures from the partial Fourier recovery problem. ACNS.

[BSS22] Boudgoust, Sakzad, and Steinfeld. Vandermonde meets Regev: Public Key Encryption Schemes Based on Partial Vandermonde Problems. DCC.

Conclusion

Conclusion on lattice-based crypto

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Thank you