Lattice-based crypto, part 1:
Algorithmic problems over lattices

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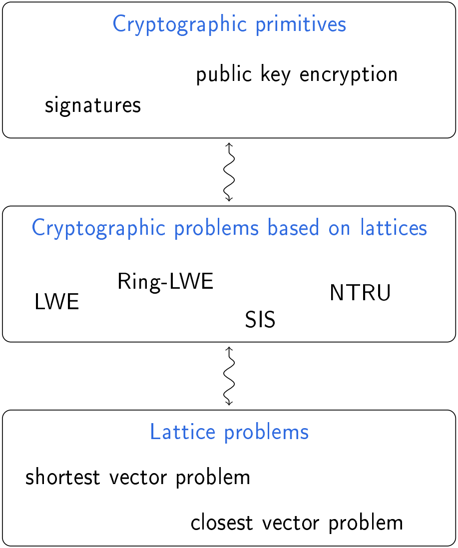
Plan of the talks



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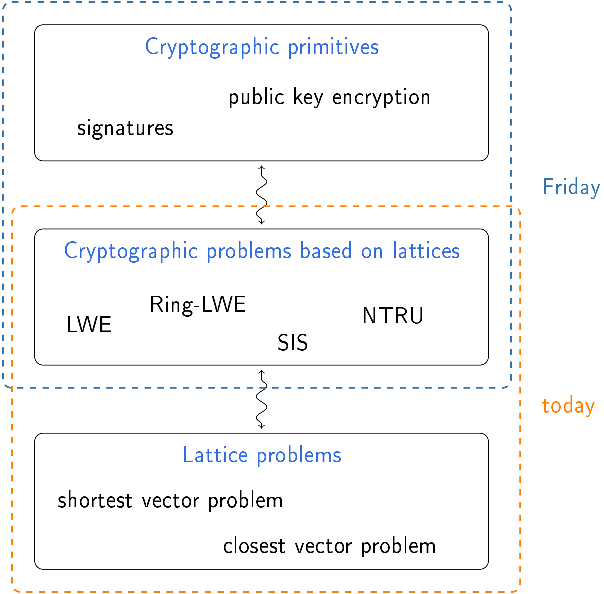
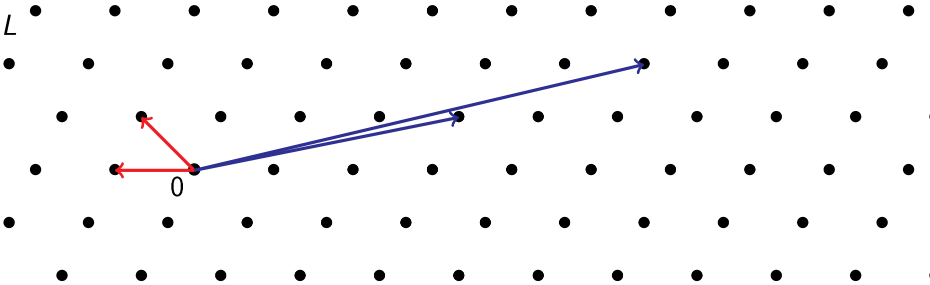


Outline of the talk

1. Lattices and lattice problems
2. Cryptographic problems based on lattices
3. Algorithms

Outline of the talk

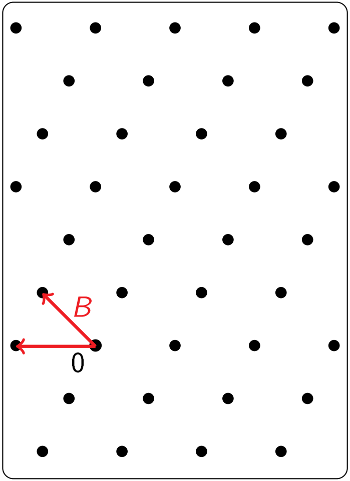
Lattices



- $L = \mathcal{L}(B) = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
- $B \in \mathrm{GL}_n(\mathbb{R})$ is a basis
- n is the dimension of L (or rank)

Representing a lattice

Representation of a lattice L:
a basis $B \in \mathbb{Z}^{n \times n}$ of L



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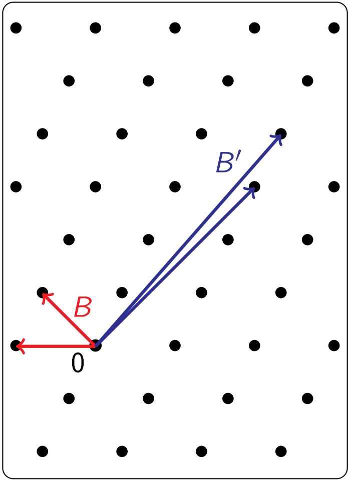
- the basis B is not unique
- some choices of B can make some algorithmic problems easier



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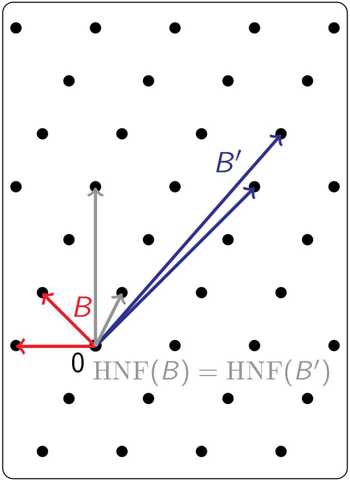
Solution: take the Hermite Normal Form (HNF) of any B

- it is unique $(\mathrm{HNF}(B) = \mathrm{HNF}(B'))$
- it is efficiently computable



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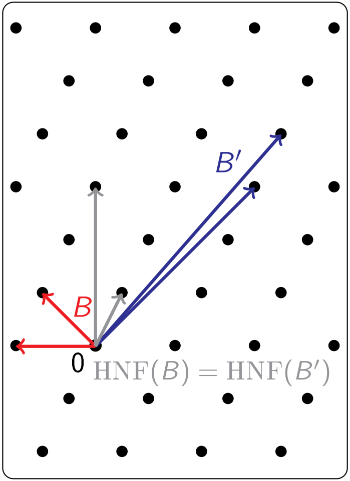
$\Rightarrow$ canonical representation of L
(i.e., worse basis ever)



Algorithmic problems on lattices

Input: the HNF basis of any lattice

Example of problems:

(1) Testing equality of lattices
(2) Testing inclusion of lattices
(3) Intersecting two lattices
(4) Computing a short vector of a lattice
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Quiz: which ones are easy or hard?

easy: polynomial time — hard: no polynomial time algorithm known

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Testing inclusion / equality

Exercise

Given $B_1, B_2 \in \mathrm{GL}_n(\mathbb{R})$, how do you test if

1. $\mathcal{L}(B_1) \subseteq \mathcal{L}(B_2)$
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$$\Leftrightarrow B_1 \cdot B_2^{-1} \in \mathbb{Z}^{n \times n}$$

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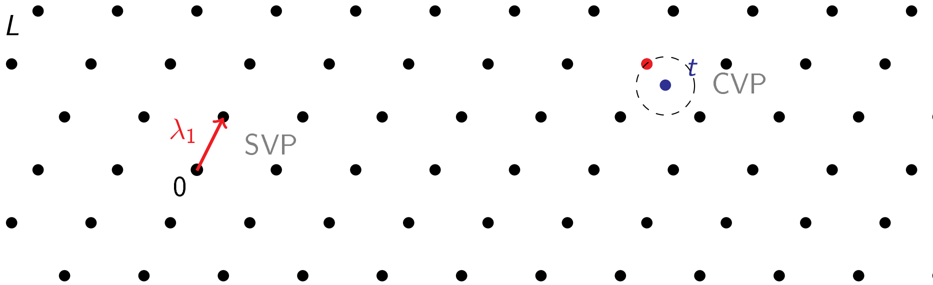
2. $\mathcal{L}(B_1) = \mathcal{L}(B_2) \Leftrightarrow \mathcal{L}(B_1) \subseteq \mathcal{L}(B_2)$ and $\mathcal{L}(B_2) \subseteq \mathcal{L}(B_1)$

(Hard) lattice problems



SVP : Shortest Vector Problem
(input: HNF basis of L)

CVP : Closest Vector Problem
(input: HNF basis of L and target t)

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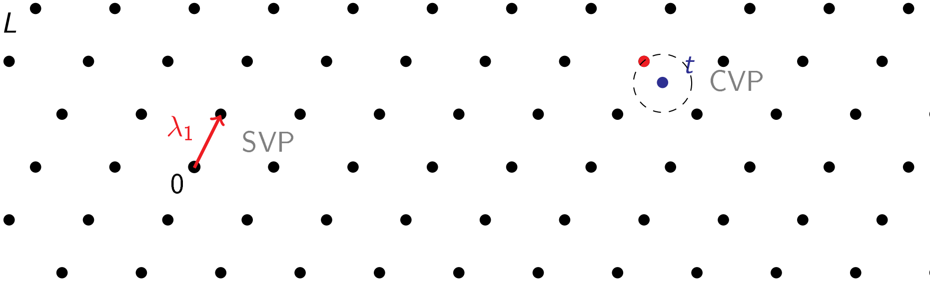

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Supposedly hard to solve when n is large
(even with a quantum computer)

How hard is SVP/CVP?

In theory: best algorithm has asymptotic complexity $2^{c \cdot n + o(n)}$
(for some $c \approx 0.292$, or $c \approx 0.265$ for quantum computers [Laa15])

$\Rightarrow$ not polynomial

[Laa15] Laarhoven. Search problems in cryptography.

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- from $n = 500$ to $n = 1000 \rightsquigarrow$ cryptography

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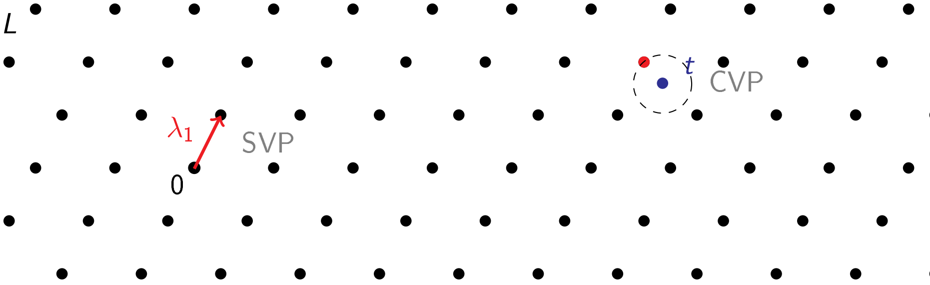
Approximate lattice problems



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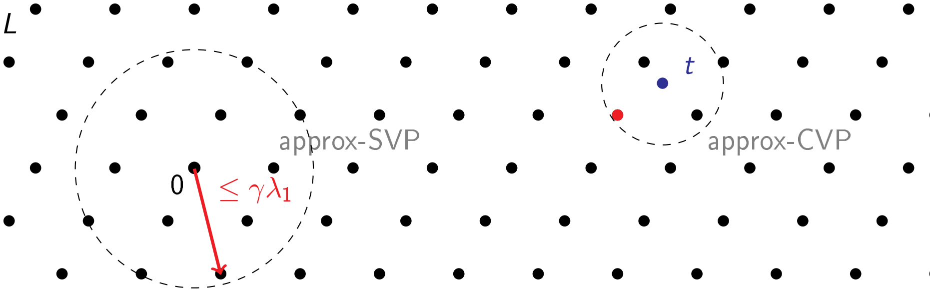


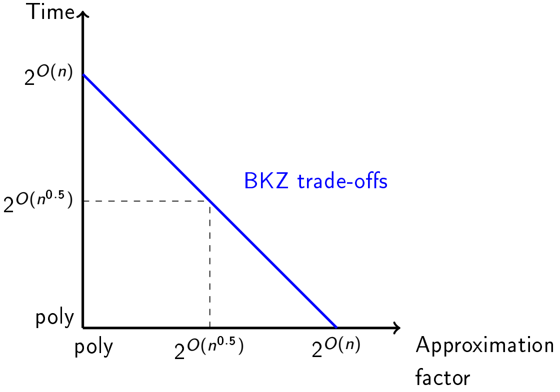
approx-SVP : Shortest Vector Problem

approx-CVP : Closest Vector Problem

Supposedly hard to solve when n is large
when the approximation factor is small ($\mathrm{poly}(n)$)
(even with a quantum computer)

Asymptotic hardness of approx-SVP/CVP

Best Time/Approximation trade-off for SVP, CVP (even quantumly):
BKZ algorithm [Sch87,SE94]



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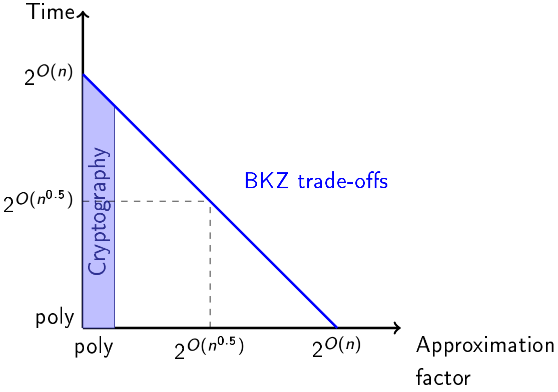

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Section's conclusion

γ-SVP and γ-CVP: (i.e., approx SVP/CVP with approx factor γ)

- best algorithm so far requires $2^{\Omega(n)}$ time if $\gamma = \mathrm{poly}(n)$
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- exact vs approx
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Zoo of lattice problems: many variants $\Rightarrow$ how do they compare?

- exact vs approx
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- one short vector vs a short basis
- see [Ste16] for a very nice picture

[Ste16] Stephens-Davidowitz. Dimension-preserving reductions between lattice problems.
http://www.noahsd.com/latticeproblems.pdf

Outline of the talk

Limitations of SVP (and CVP)

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For crypto, we need problems that are hard on average

(i.e., for a random instance, the problem is hard with overwhelming probability)

SIS

The SIS problem

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

SIS (Short Integer Solution) [Ajt96]

Parameters: B and q

Problem: Given $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{m \times n})$ (with $n \log q < m$)

Find $x \in \mathbb{Z}^m$ s.t. $x^T A = 0 \bmod q$ with $\|x\| \leq B$ and $x \neq 0$.

[Ajt96] Ajtai. Generating hard instances of lattice problems. STOC.

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Solving SIS with non-negligible probability	$\gtrsim$	Solving approx-SVP in any lattice of rank n

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SIS is as hard as worst-case lattice problems

Theorem [Ajt96]

For any $m = \mathrm{poly}(n)$ and $B > 0$ and sufficiently large $q \geq B \cdot \mathrm{poly}(n)$, there is a reduction from solving SIS to solving γ-SIVP on arbitrary n-dimensional lattice, for some approximation factor $\gamma = B \cdot \mathrm{poly}(n)$.

(SIVP = shortest independent vectors problems.

Objective: find n short linearly independent vectors in the lattice)

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- the poly quantities have been improved in more recent works
- see [Pei16] for a survey

[Pei16] Peikert. A decade of lattice cryptography. Foundations and trends in theoretical computer science

SIS is a lattice problem

SIS (Short Integer Solution)

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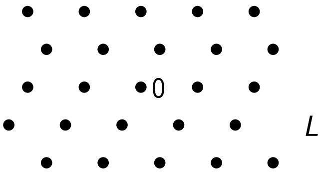

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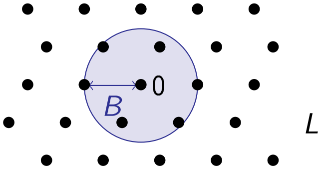

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SIS $\approx$ approx-SVP in L

LWE

The LWE problem

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

LWE (Learning With Errors) [Reg05]

Sample $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{m\times n})$ and s, $e \leftarrow \text{Uniform}(\{-B, \cdots, B\}^n)$

Given A and b, where $b := A s + e \bmod q$

Recover s or e

[Reg05] Regev. On lattices, learning with errors, random linear codes, and cryptography. STOC.

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Remark. Sometimes s is uniform in $\mathbb{Z}_q$ (not small)

- this is (almost) equivalent
- prove it (*hint:* you are allowed to change m)

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Solving LWE with non-negligible probability $\gtrsim$ (quantumly!) Solving approx-SVP in any lattice of rank n

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Solving approx-SVP in any lattice of rank m	$\gtrsim$	Solving LWE with non-negligible probability	$\gtrsim$ quantumly!	Solving approx-SVP in any lattice of rank n

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LWE is quantumly as hard as worst-case lattice problems

Theorem [Reg05]

For any $m = \mathrm{poly}(n)$, modulus $q \leq 2^{\mathrm{poly}(n)}$ and $B \geq 2\sqrt{n}$, there is a quantum reduction from solving LWE to solving γ-SIVP on arbitrary n-dimensional lattice, for some approximation factor $\gamma = \tilde{O}(n \cdot q/B)$.

⚠ the reduction is for a variant of LWE where s and e are sampled from a discrete Gaussian distribution of parameter B ⚠

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Remark: the reduction can be made fully classical [Pei09, BLPRS13]

[Pei09] Peikert. Public-key cryptosystems from the worst-case shortest vector problem. STOC.

[BLPRS13] Brakerski, Langlois, Peikert, Regev, and Stehlé. Classical hardness of learning with errors. STOC

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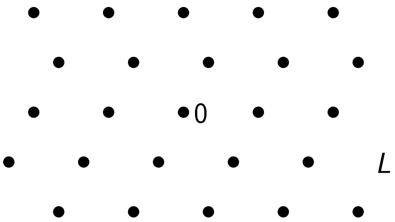

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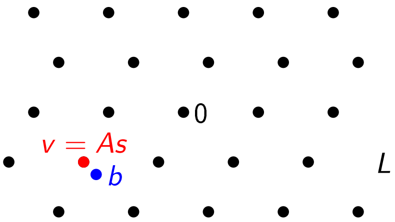

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$$b = v + e,$$

where $v \in L$ and e small

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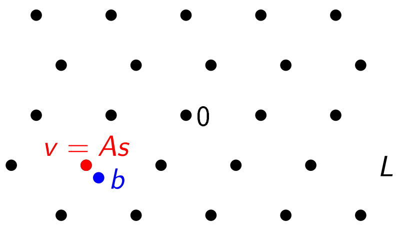

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SIS $\overset{\sim}{\longleftrightarrow}$ average case SVP

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Decision variant of LWE

decision-LWE

Sample $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{n\times m})$ and s, $e \leftarrow \text{Uniform}(\{-B, \cdots, B\}^n)$

Given A and b, where

$$b := A s + e \bmod q \quad \text{or} \quad b \leftarrow \text{Uniform}(\mathbb{Z}_q^n)$$

Guess whether b is uniform or not.

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$\Rightarrow$ decision problems can be easier to use for crypto

LWE vs SIS

decision-LWE $\overset{\sim}{\Longleftrightarrow}$ (search) LWE $\underset{\text{quantum}}{\overset{\sim}{\Longleftrightarrow}}$ SIS

LWE vs SIS

$$\text{decision-LWE} \overset{\sim}{\Longleftrightarrow} \text{(search) LWE} \underset{\text{quantum}}{\overset{\sim}{\Longleftrightarrow}} \text{SIS}$$

Exercise

Prove that decision-LWE $\leq$ SIS

Hint: Assume that we know x^T small such that $x^T A = 0 \bmod q$

How to distinguish

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Solution: Compute $x^T b$, this gives

$$\underbrace{x^T A}_{0} s + x^T e = \underbrace{x^T e}_{\text{small}} \bmod q \quad \text{vs} \quad \underbrace{x^T b}_{\text{uniform}}$$

Section's summary

(decision) LWE / SIS:

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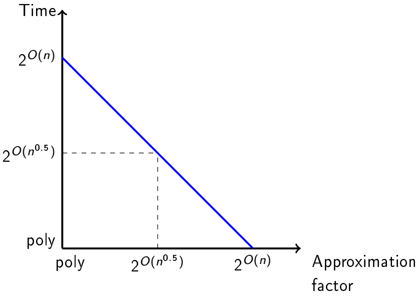
- very useful survey [Pei16]

[Pei16] Peikert. A decade of lattice cryptography. Foundations and trends in theoretical computer science

Outline of the talk

Various SVP algorithms

BKZ trade-offs



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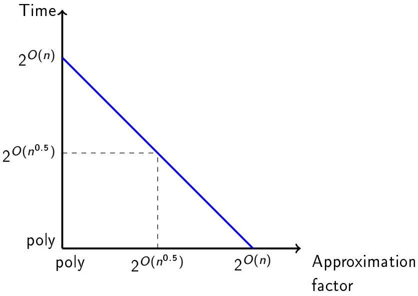

Lagrange-Gauss algorithm: dim 2

- exact SVP
- polynomial time

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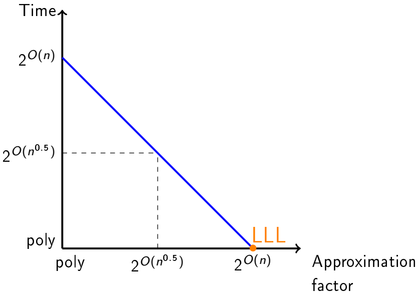

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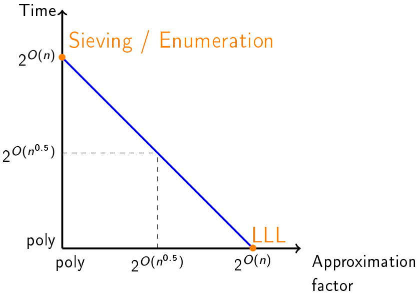

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Sieving algorithm: dim n

- exact SVP
- time $2^{O(n)}$

Lagrange-Gauss algorithm

video

Lagrange-Gauss algorithm

video

Theorem: the algorithm

- finds a shortest vector of L
- runs in polynomial time

The LLL algorithm [LLL82]

Input: basis $B = (b_1, \ldots, b_n)$

[LLL82] Lenstra, Lenstra, and Lovász. Factoring polynomials with rational coefficients. Mathematische annalen.

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This algorithm

- finds $v \in L$ with $\|v\|_2 \leq 2^n \cdot \lambda_1(L)$
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The LLL algorithm [LLL82]

Input: basis $B = (b_1, \ldots, b_n)$

Main idea: improve the basis locally on blocks of dimension 2
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Algorithm:

- while there exist i such that $\|b_i\|_2 > 4/3 \cdot \lambda_1(L_i)$
  (L_i is roughly the lattice spanned by (b_i, b_{i+1}))
  - run Lagrange-Gauss on L_i

This algorithm

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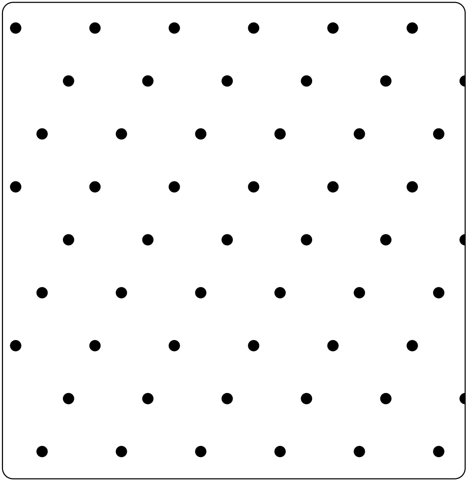
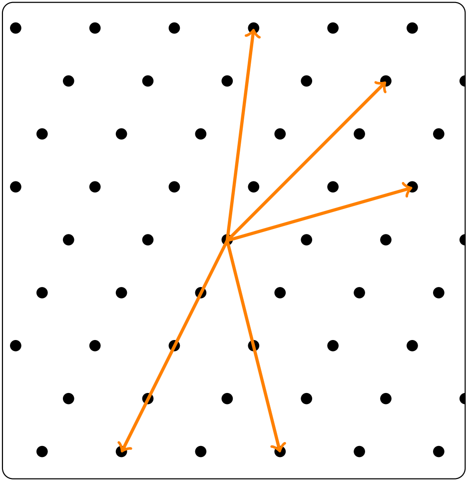
Sieving algorithm [AKS01]



Sieving:

[AKS01] Ajtai, Kumar, and Sivakumar. A sieve algorithm for the shortest lattice vector problem. STOC

Sieving algorithm [AKS01]



Sieving:

- Create many large vectors

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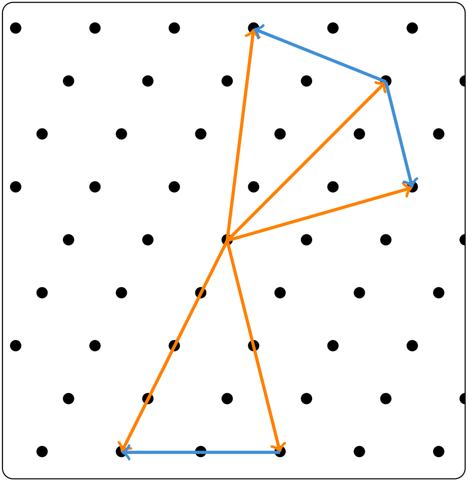

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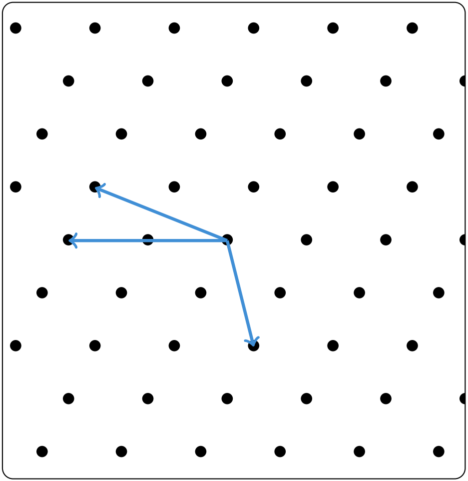

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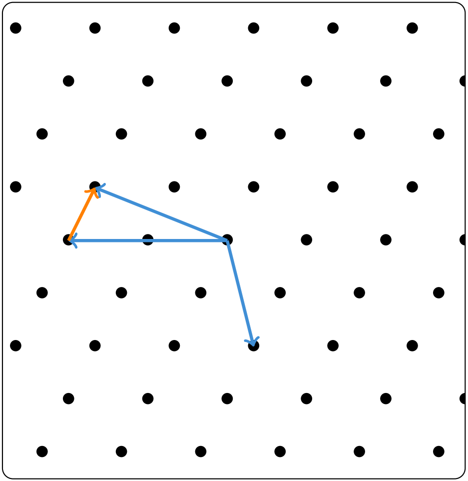

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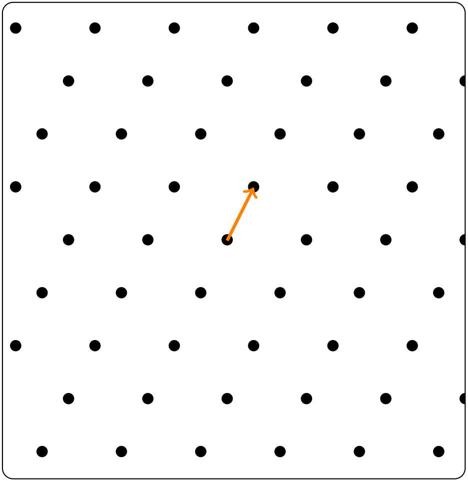

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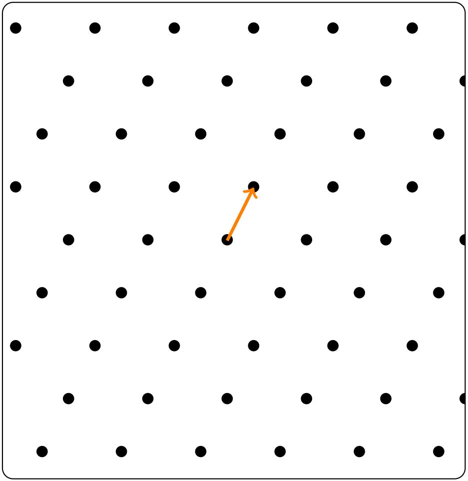

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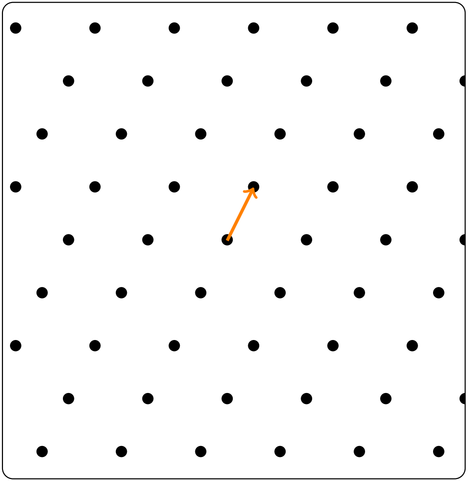

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- finds a shortest vector
- runs in time $2^{O(n)}$

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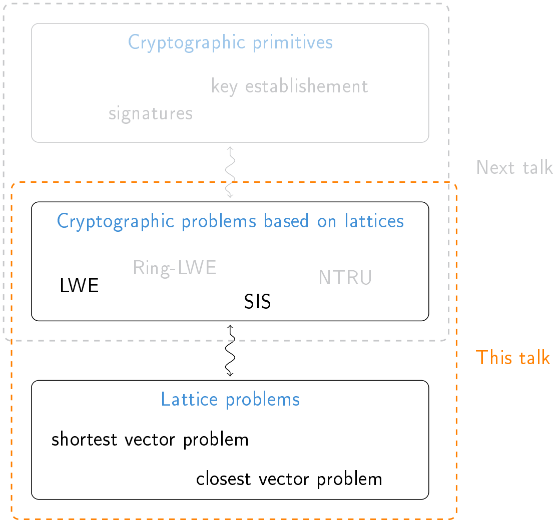
Conclusion

What we have seen



Takeaway: all these problems are supposed to be quantumly hard

(for a good choice of parameters)