TD 14 – Random walks

Exercice 1. *Cover time in graphs*

Given a finite, undirected non-bipartite and connected graph $G = (V, E)$, recall that the *cover time* of G is the maximum over all vertices $v \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from v.

1. Recall that $h_{v,u}$ is the expected number of steps to reach u from v and $h_{u,u} = \frac{2|E|}{d(u)}$. Show that

$$\sum_{w \in N(u)} (1 + h_{w,u}) = 2|E|.$$

2. Let T be a *spanning tree* of G (i.e. T is a tree with vertex set V). Show that there is a *tour* (i.e. a walk with the same starting and ending points) passing each edge of T exactly twice, once for each direction.
3. Let $v_0, v_1, ..., v_{2|V|-2} = v_0$ be the sequence of vertices of such tour. Prove that

$$\sum_{i=0}^{2|V|-3} h_{v_i,v_{i+1}} < 4|V| \times |E|.$$

4. Conclude that the cover time of G is upper-bounded by $4|V| \times |E|$.

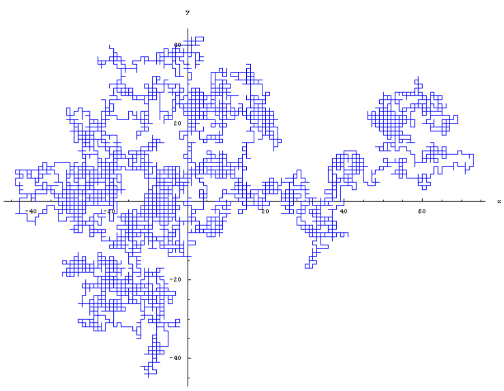
Exercice 2. *$s - t$ connectivity in graph*

Suppose we are given a finite, undirected and non-bipartite graph $G = (V, E)$ and two vertices s and t in G. Let $n = |V|$ and $m = |E|$. We want to determine if there is a path connecting s and t. This is easily done in linear time using a standard breadth-first search or depth-first search. Such algorithms, however, require $\Omega(n)$ space. Here we develop a randomized algorithm that works with only $O(\log n)$ bits of memory. This could be even less than the number of bits required to write the path between s and t. The algorithm is simple :
Perform a random walk from s on G. If the walk reaches t within $4n^3$ steps, return that there is a path. Otherwise, return that there is no path.

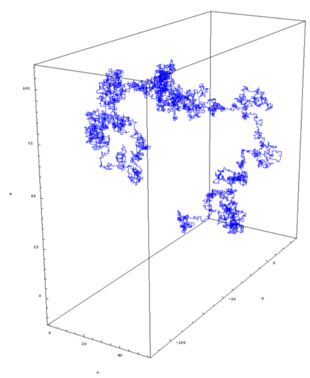
1. Show that the algorithm returns the correct answer with probability at least 1/2, and it only makes errors by returning that there is no path from s to t when there is such a path.

Exercice 3. *Random walks on $\mathbb{Z}^d$*

Consider the simple random walk over $\mathcal{Z}^d$ with probability $1/2d$ to jump towards any of the $2d$ neighbors in the grid. The walk is clearly irreducible.



A random walk in $\mathcal{Z}^2$
(10000 steps, Wikipedia)



A random walk in $\mathcal{Z}^3$
(10000 steps, Wikipedia)

1. For $d = 1$, whether it is recurrent? positive recurrent?
2. For $d = 2$, whether it is recurrent? positive recurrent?
   Hint : Consider decomposing the walk into two independent walks.
3. In the case $d = 3$, for every n, show that

$$\mathbb{P}(S_{2n} = 0) = \sum_{r+s+t=n} \binom{2n}{n} \binom{n}{r,s,t}^2 \frac{1}{6^{2n}},$$

   where S_i is the location of the walk at time i.
4. Show that

$$\sum_{n=0}^{\infty} \mathbb{P}(S_{2n} = 0) < \infty$$

   and conclude for the case $d = 3$.

Exercice 4. *Cat and mouse*

A cat and mouse each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G.

1. Show an upper bound of $\mathcal{O}(m^2 n)$ on the expected time before the cat eats the mouse. (Hint : Consider a Markov chain whose states are the ordered pairs (a, b), where a is the position of the cat and b is the position of the mouse.)