Introduction to lattice-based cryptography

Part II

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PEPR PQ-TLS summer school, Anglet

Plan

Part I

Lattice problems

- Lattices
- Hard problems

Algorithms for lattice problems

- LLL, sieving, BKZ

Constructions

- Public key encryption
- Signatures

Part II

NTRU and signature

- NTRU, Falcon

LWE and public key encryption

- LWE
- Regev's encryption, Kyber

Algebraic Lattices

- Ideal and module lattices
- NTRU, RLWE, mod-LWE

Reductions between problems

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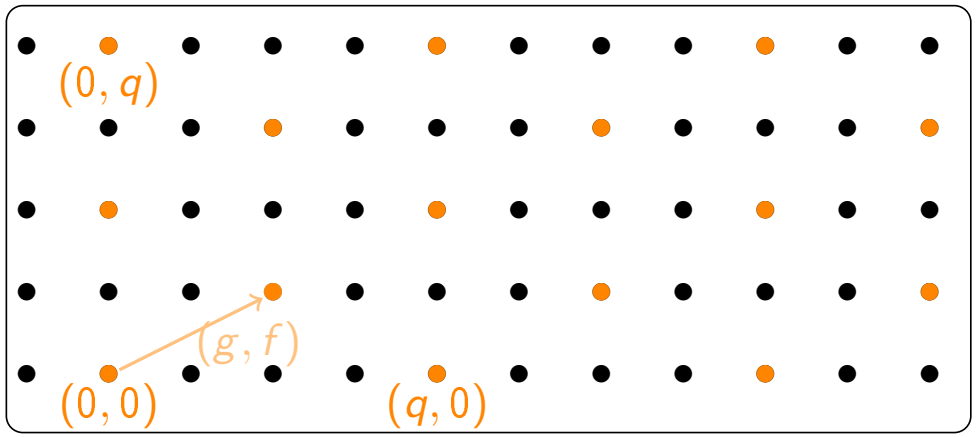
Episode 2: a random hard lattice $\mathcal{L}$ (+ a short basis) $\Rightarrow$ signature and PKE

How do we generate a hard lattice (+ a short basis)?

NTRU and Falcon's signature scheme

(one of the three 2022 NIST standard)



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- 2023 call for additional digital signatures ("not lattice-based")
  - 40 submissions, still under consideration

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NTRU [HPS98]

Parameters:
- q prime and large (e.g., $q = 16411$)
- $B \ll \sqrt{q}$ integer (e.g., $B = 5$)

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  (e.g., $f = -2$, $g = 3$)
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- we should replace integers by polynomials (degree 512 or 1024)
- for the moment, let's keep integers (and pretend it is hard)

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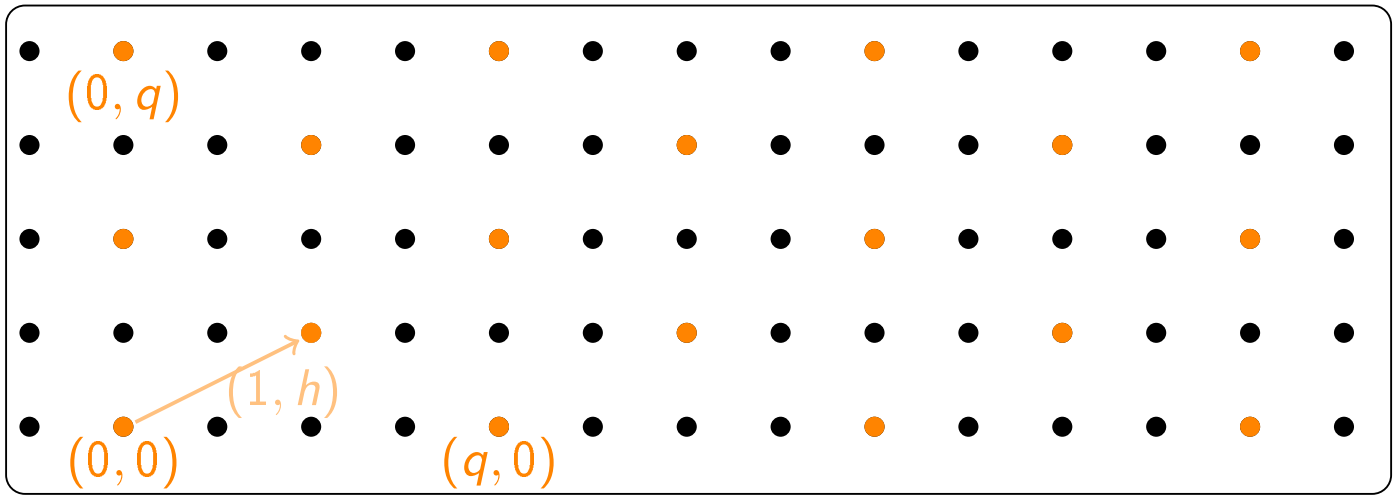

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no adversary can compute a short basis of $\mathcal{L}_h$ $\impliedby$ NTRU assumption holds

NTRU: wrapping up

We have seen: under the NTRU assumption, the following algorithm produces random hard lattices.

- sample random f, g in $\{-B, \dots, B\}$
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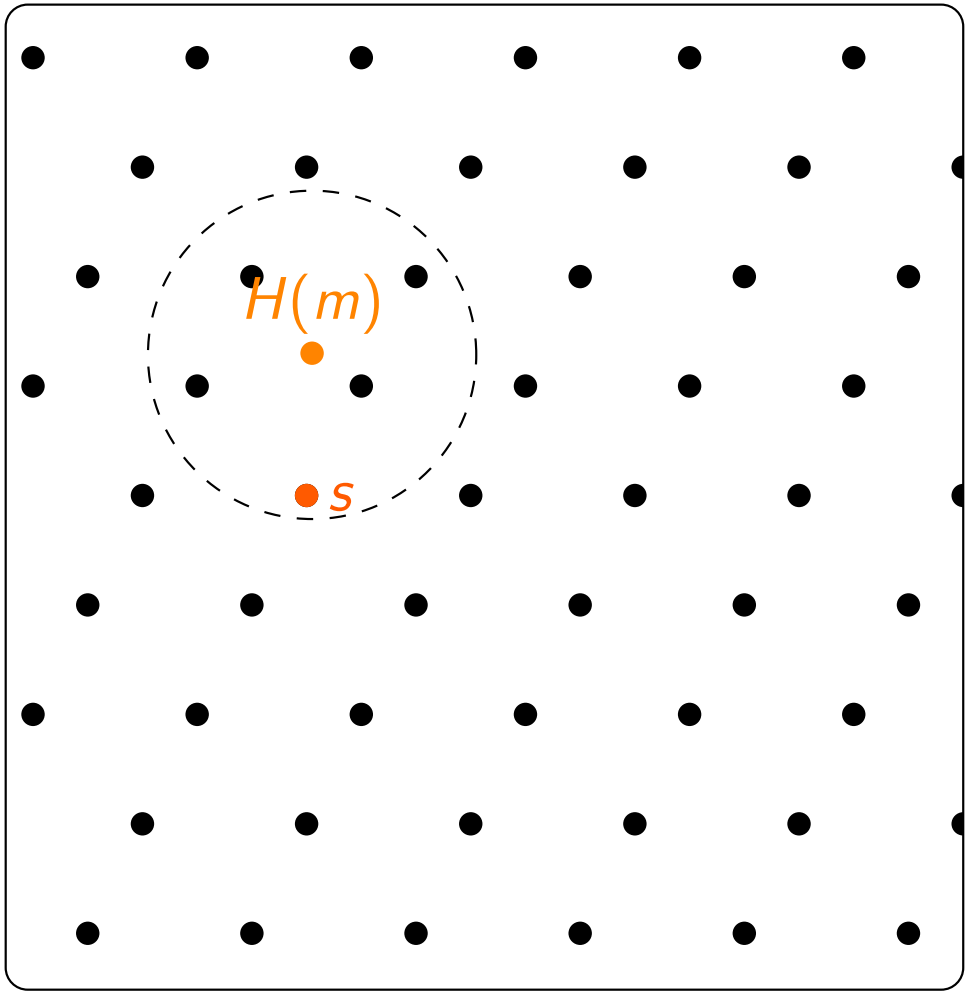
This is because we work with integers instead of polynomials (for simplicity)

$\Rightarrow$ if f, g and h are polynomials of degree d, $\mathcal{L}_h$ has dimension $2d$ instead of 2

Remember: hash-and-sign signatures



KeyGen:

- pk = bad basis of $\mathcal{L}$
- sk = short basis of $\mathcal{L}$

Sign(m, sk):

- $x = H(m)$ (hash the message)
- sample $s \in \mathcal{L} \cap \mathcal{B}_r(x)$ (small radius r)

Verify(m, s, pk):

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

Lemma: if an adversary can forge signatures, then she can recover a short basis of $\mathcal{L}$ using only pk (in the ROM)

Falcon signature scheme

Falcon signature:

Hash-and-sign signature

+

hard lattice from NTRU assumption

with polynomials of degree $d = 512$ or 1024 instead of integers

Falcon is one of the three post-quantum signature scheme standardized by the NIST in 2022.

[Falcon] Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Prest, Ricosset, Seiler, Whyte, Zhang. NIST standard

Falcon's performances

			Post-quantum standards (NIST 2022)		
	RSA	EdDSA	**Falcon**	Dilithium	SPHINCS+
Public key size (bytes)	256	32	**897**	1312	32
Signature size (bytes)	256	64	**666**	2420	17088
Signature time (µs)	665	51	**241**	208	35584
Verification time (µs)	19	142	**52**	74	2091

- Comparison for $\approx$ 128-bits security
- Timings for a processor of 1.6 GHz
  - RSA and EdDSA are obtained experimentally with openssl on my laptop
  - Falcon, Dilithium and SPHINCS+ are obtained from the number of cycles provided on their websites (assuming a 1.6 GHz processor)
- for SPHINCS+, this is the variant using SHAKE hash function

Pros and cons of Falcon

Falcon signature:

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- KeyGen and Sign are complex to implement
  - can lead to bugs in implementation

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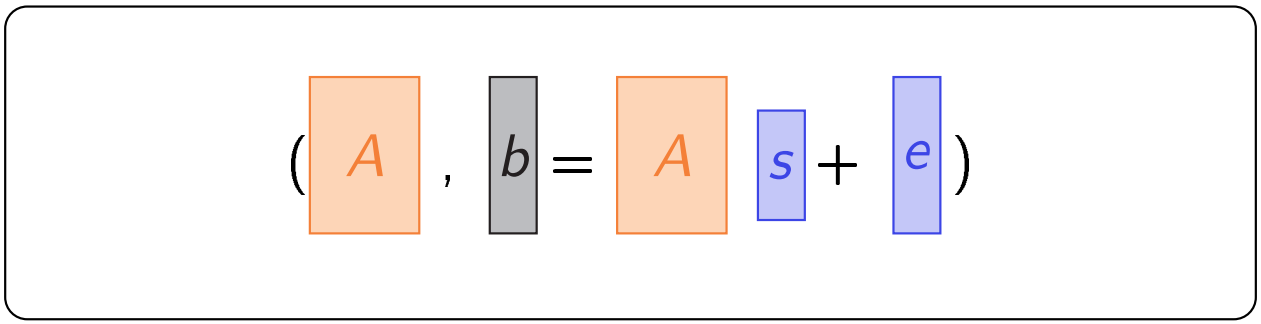
LWE and Regev's encryption scheme

(and a little bit of Kyber, another 2022 NIST standard)



The LWE problem [Reg05]

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

LWE (Learning With Errors) [Reg05]

Sample $A \leftarrow \text{Uniform}(\mathbb{Z}_q^{m \times n})$ and $s, e \leftarrow \text{Uniform}(\{-B, \cdots, B\}^n)$

Given A and b, where $b := As + e \bmod q$

Recover s or e

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LWE assumption: there is no efficient algorithm solving LWE

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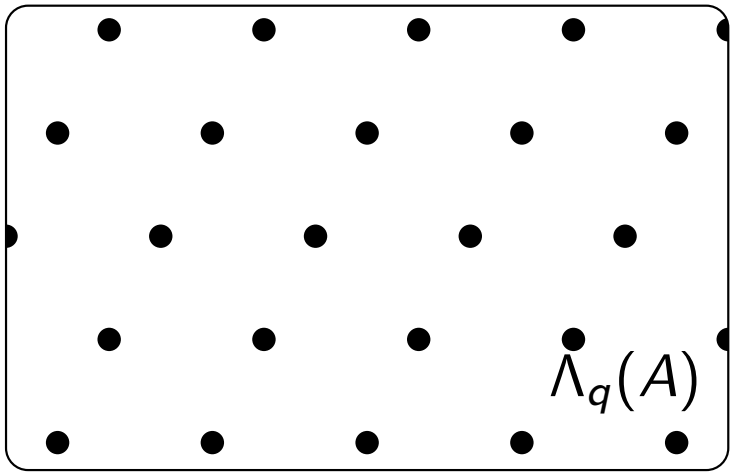

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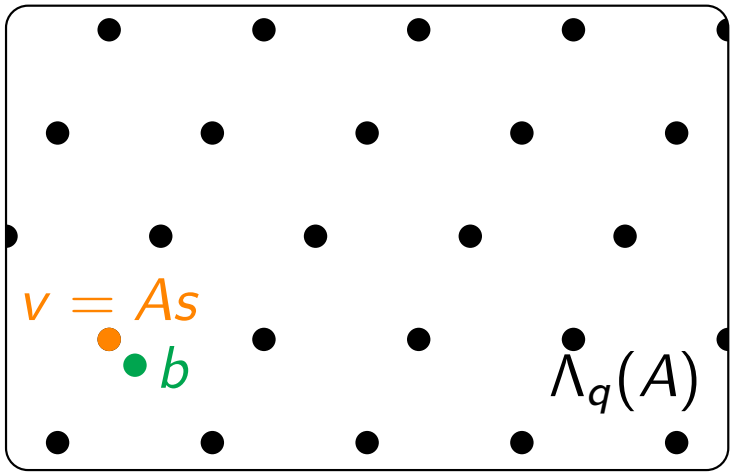

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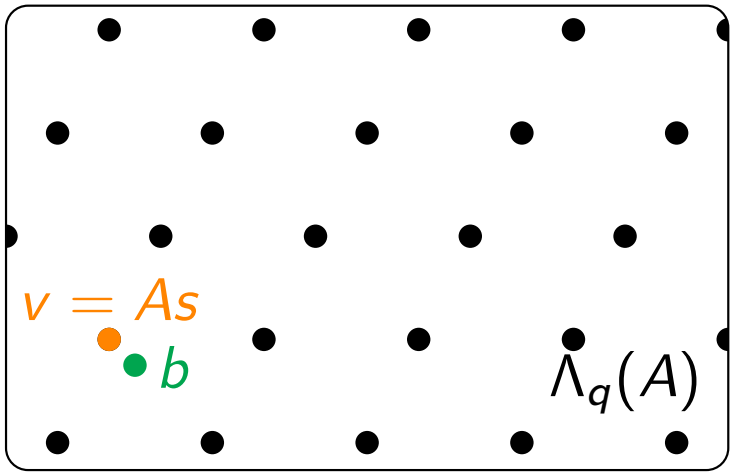

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LWE $\approx$ CVP in $\Lambda_q(A)$

Decision variant of LWE

decision-LWE

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$\Rightarrow$ decision problems can be easier to use for crypto

Regev-like encryption scheme [LPS10,LP11]

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Encrypt: message $m \in \{0, 1\}$

sample $\tilde{s}^T, \tilde{e}^T \leftarrow U_B^n$ and $e' \leftarrow U_B$

return $c = (\tilde{s}^T A + \tilde{e}^T, \tilde{s}^T b + e' + m \cdot \lfloor q/2 \rceil)$

(all $\bmod q$)

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(all mod q)

Decrypt: $c = (c_1{}^T, c_2)$

compute $r = c_1{}^T s - c_2 \bmod q$ ($r \in [0, q]$)

return 1 if r is in $[q/4, 3q/4]$ and 0 otherwise

[LPS10] Lyubashevsky, Palacio, Segev. Public-key cryptographic primitives provably as secure as subset sum. TCC.

[LP11] Lindner and Peikert. Better key sizes (and attacks) for LWE-based encryption. CT-RSA.

Correctness

Theorem

If $q \geq 8 \cdot n \cdot B^2 + 4 \cdot B$, then the scheme is correct.

Correctness: for any message m and any $(pk, sk) \leftarrow \mathsf{KeyGen}$, it holds that

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Proof: on the board

Decryption failures: if we replace U_B by a Gaussian distribution, the scheme might fail with very small probability (s and e might have coefficients $\geq B$)

Security: high level

Public information:

A

$b = As + e$

$c_1^T = \tilde{s}^T A + \tilde{e}^T$

$c_2 = \tilde{s}^T b + e' + m \cdot \lfloor q/2 \rceil$

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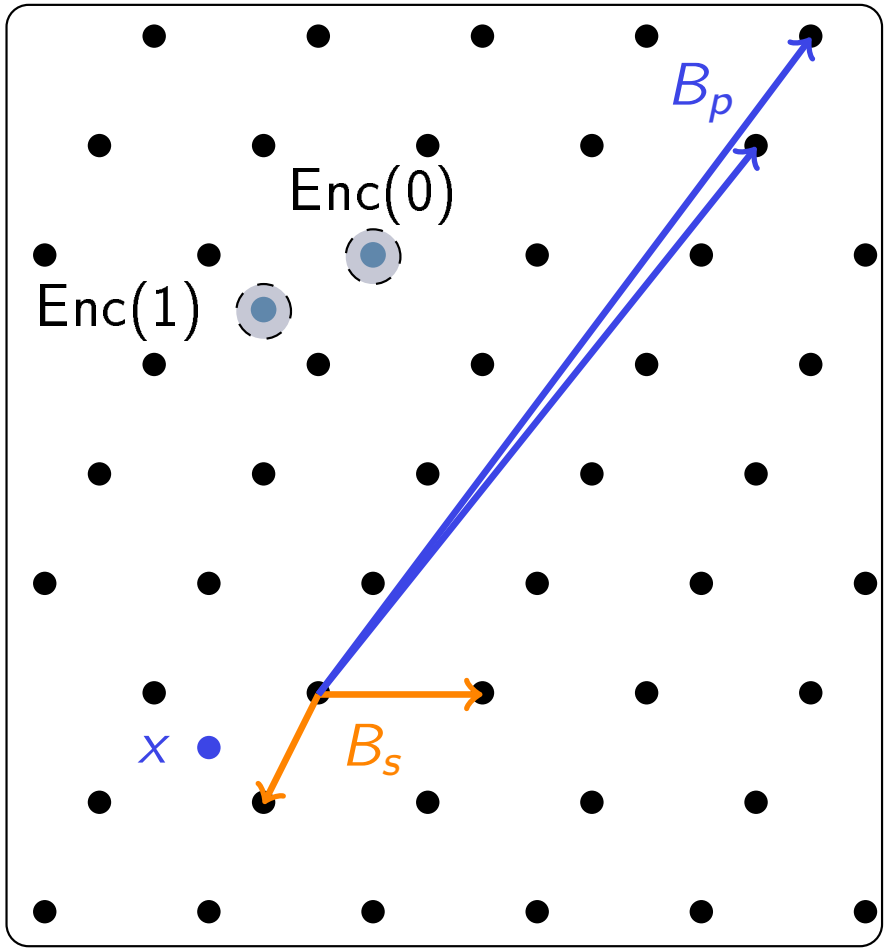
What is the relation between

KeyGen:
- $sk = s$
- $pk = (A, b = As + e)$

Encrypt: $c = (c_1^T, c_2)$ with
- $c_1^T = \tilde{s}^T A + \tilde{e}^T$
- $c_2 = \tilde{s}^T b + e' + m \cdot \lfloor q/2 \rceil$

Decrypt:
- compute $r = c_1^T s - c_2 \bmod q$
- return $\begin{cases} 1 & \text{if } r \in [q/4, 3q/4] \\ 0 & \text{otherwise} \end{cases}$



Kyber

Kyber key exchange:

Regev-like public key encryption

(in its KEM version)

+

module variant of LWE

(see next part)

Kyber is the only post-quantum key encapsulation mechanism standardized by the NIST in 2022.

[Kyber] Avanzi, Bos, Ducas, Kiltz, Lepoint, Lyubashevsky, Schanck, Schwabe, Seiler, Stehlé. NIST standard

Summary on NTRU and LWE

NTRU and LWE are average-case problems

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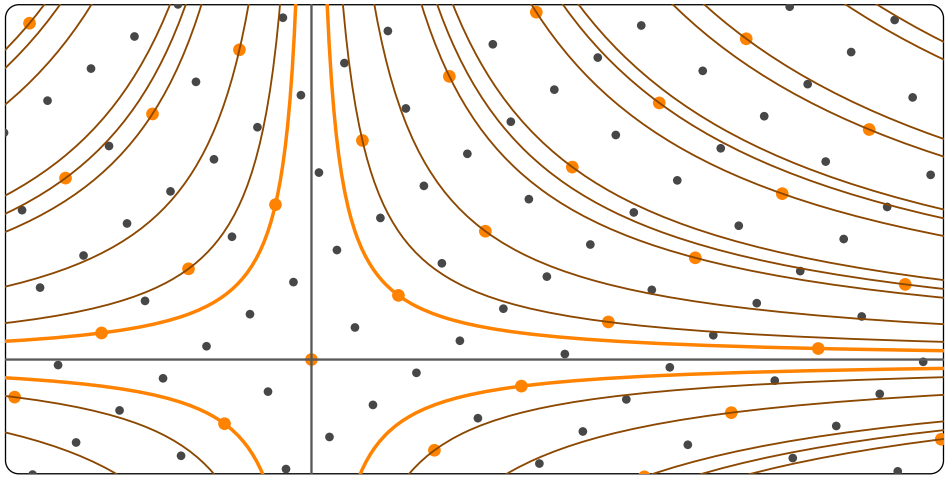
We have seen:
- signature based on NTRU
- encryption based on LWE

We can also do:
- signature based on LWE
- encryption based on NTRU

Algebraic lattices



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Number field: $K = \mathbb{Q}[X]/P(X)$ (P irreducible, $\deg(P) = d$)

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Ring of integers: $\mathcal{O}_K \subset K$, for this talk $\mathcal{O}_K = \mathbb{Z}[X]/P(X)$

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Embeddings

$(K = \mathbb{Q}[X]/P(X), \quad \alpha_1, \cdots, \alpha_d$ complex roots of $P(X))$

Coefficient embedding: $\Sigma : \begin{array}{rcl} K & \to & \mathbb{R}^d \\ \sum_{i=0}^{d-1} y_i X^i & \mapsto & (y_0, \cdots, y_{d-1}) \end{array}$

Canonical embedding: $\sigma : \begin{array}{rcl} K & \to & \mathbb{C}^d \\ y(X) & \mapsto & (y(\alpha_1), \cdots, y(\alpha_d)) \end{array}$

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Which embedding should we choose?

- coefficient embedding is used for constructions (efficient implementation)
- canonical embedding is used in cryptanalysis / reductions (nice mathematical properties)
- for fields used in crypto, both geometries are $\approx$ the same

Ideals

Ideal: $I \subseteq \mathcal{O}_K$ is an ideal if

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- $I_1 = \{2a \,|\, a \in \mathbb{Z}\}$ and $J_1 = \{6a \,|\, a \in \mathbb{Z}\}$ in $\mathcal{O}_K = \mathbb{Z}$
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Ideal lattices

$\mathcal{O}_K$ is a lattice:

- $\mathcal{O}_K = 1 \cdot \mathbb{Z} + X \cdot \mathbb{Z} + \cdots + X^{d-1} \cdot \mathbb{Z}$
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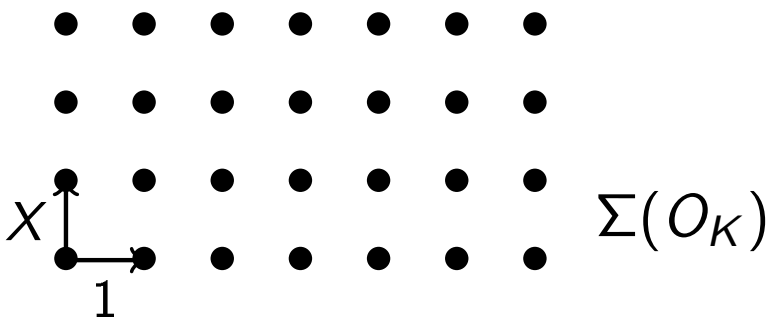
(this is also true for non principal ideals)

(we can replace Σ by σ and $\mathbb{Z}^d$ by $\mathbb{C}^d$)

Ideal lattices (2)



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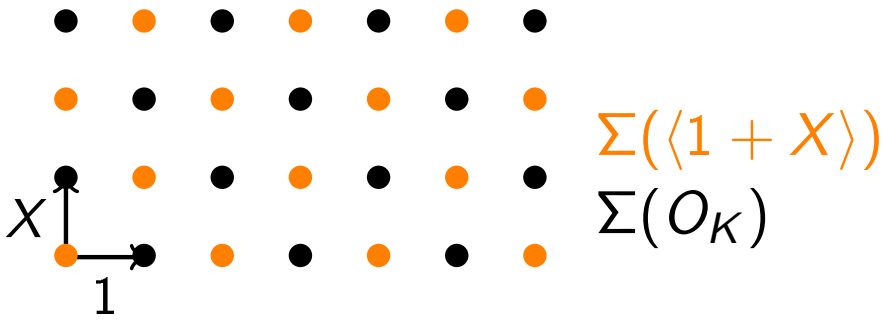
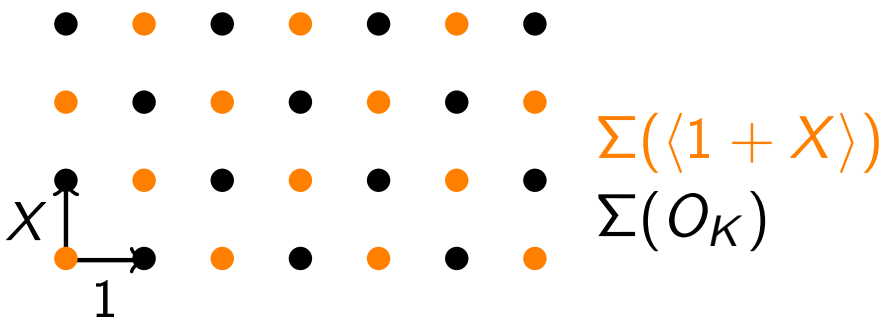


Ideal lattices (2)

Basis of $\langle g \rangle$: $g, g \cdot X, \cdots, g \cdot X^{d-1}$



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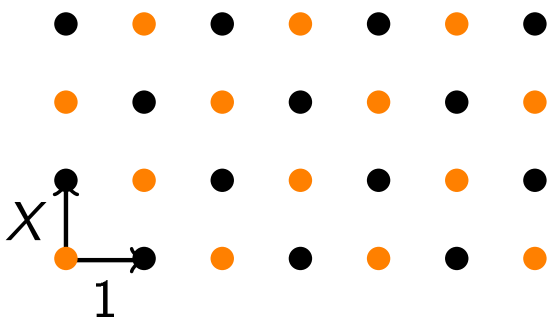

$\Sigma(\langle 1 + X \rangle)$

$\Sigma(O_K)$

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(in $K = \mathbb{Q}[X]/X^d + 1$)

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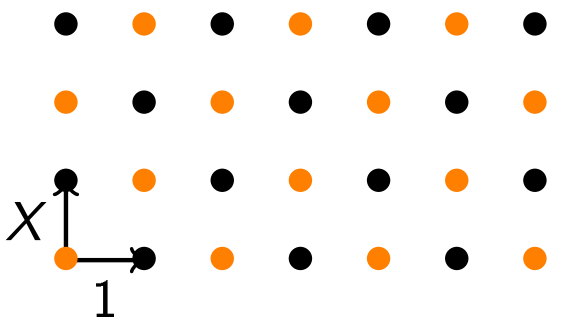

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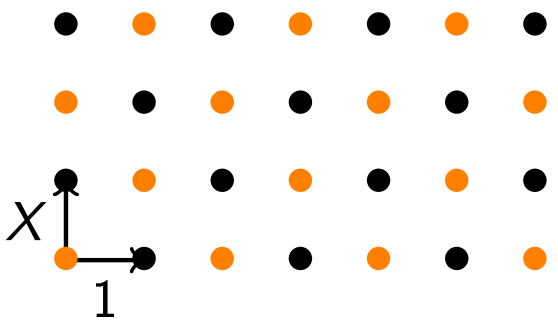

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Module lattices

(Free) module:

$$M = \{B \cdot x \,|\, x \in {\mathcal{O}_K}^k\} \text{ for some matrix } B \in {\mathcal{O}_K}^{k \times k} \text{ with } \det_K(B) \neq 0$$

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  (if the module is not free, it has a "pseudo-basis" instead)

$\Sigma(M)$ is a lattice:

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- with basis $(\Sigma(b_i X^j))_{\substack{1 \leq i \leq k \\ 0 \leq j < d}}$ (b_i columns of B)

Modules vs ideals

Ideal	$=$	Module of rank 1
(principal ideal	$=$	free module of rank 1)

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Ideal	=	Module of rank 1
(principal ideal	=	free module of rank 1)

In $K = \mathbb{Q}[X]/(X^d + 1)$:

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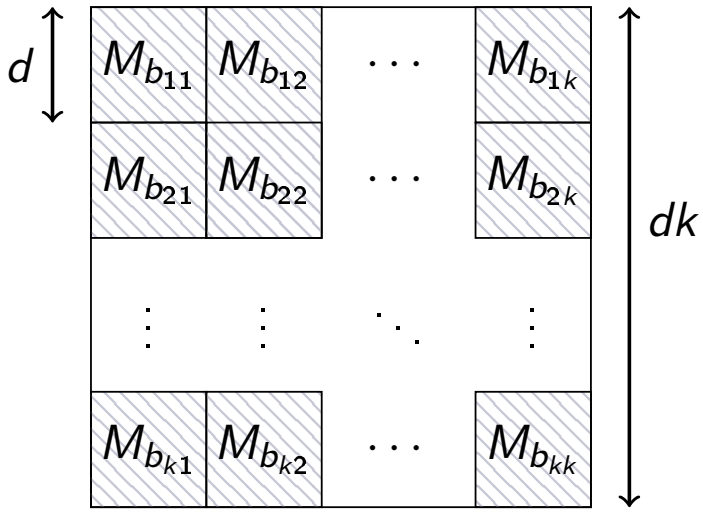
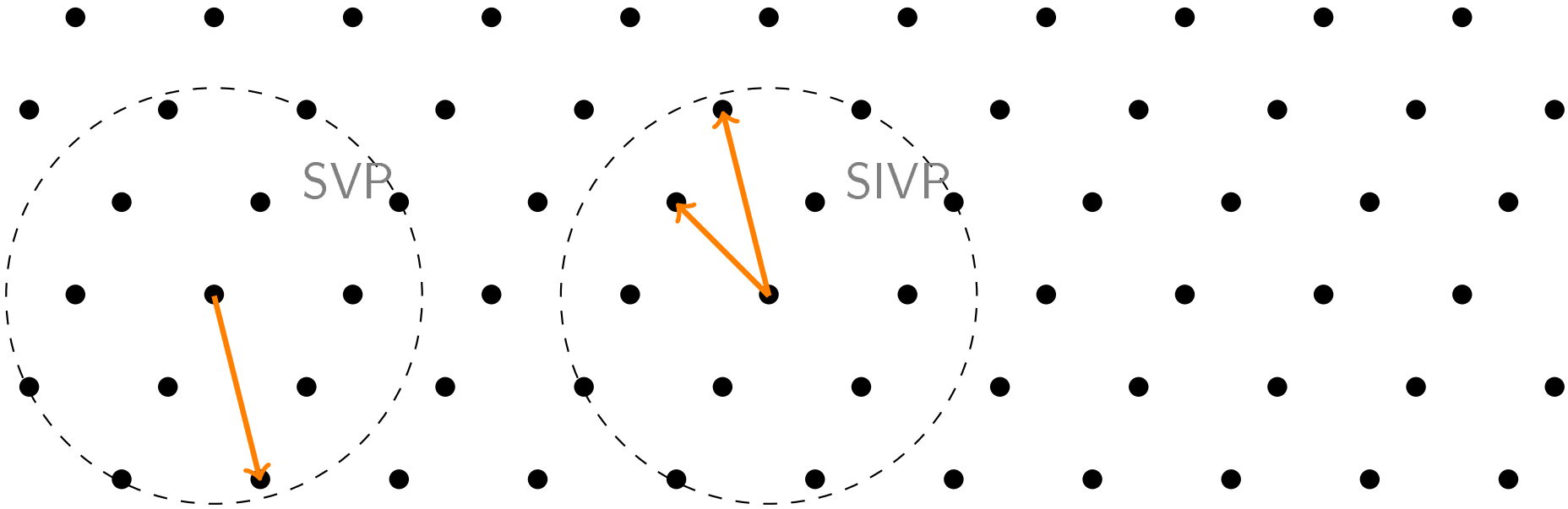
basis of a
principal ideal lattice



basis of a free module lattice
of rank k

Algorithmic problems



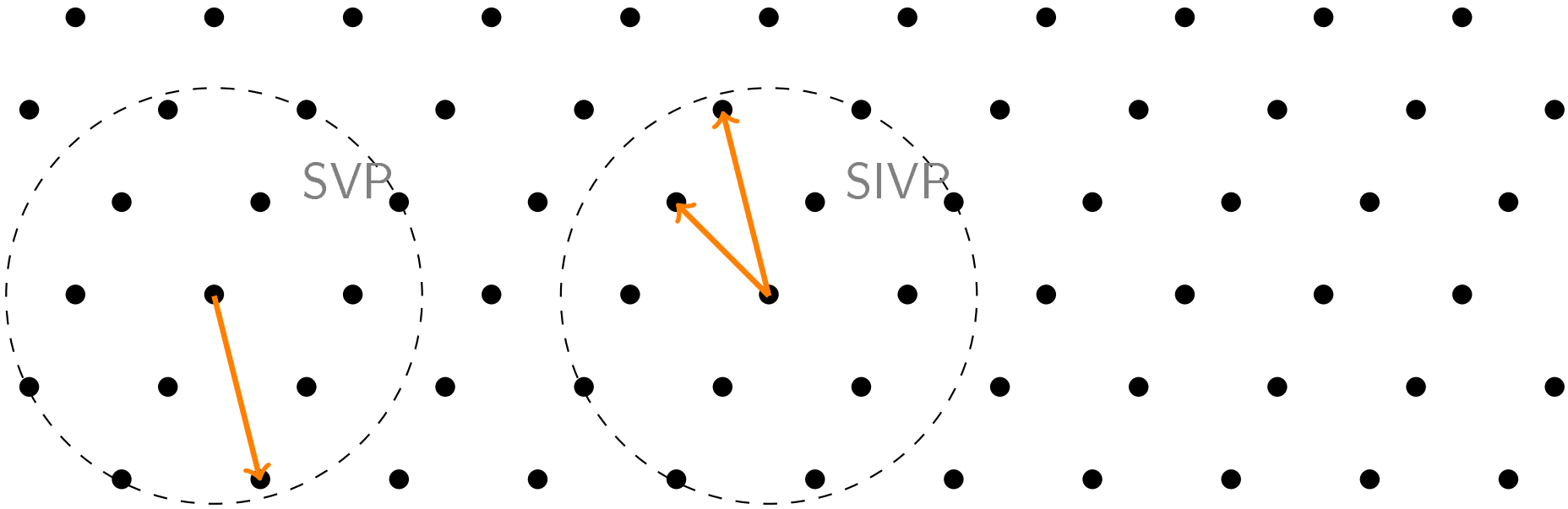
γ-SVP

shortest vector problem

γ-short basis

shortest basis problem

Algorithmic problems



γ-SVP
shortest vector problem

γ-short basis
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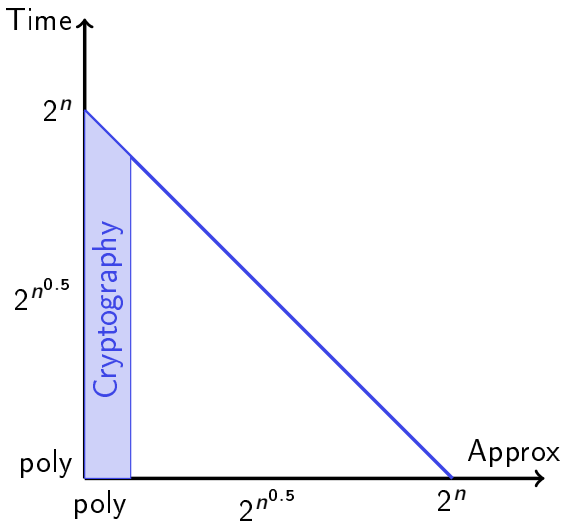
Notations:

- id-X = problem X restricted to ideal lattices
- mod-X_k = problem X restricted to module lattices of rank k

(worst-case: we want algorithms for all ideal/module lattices)

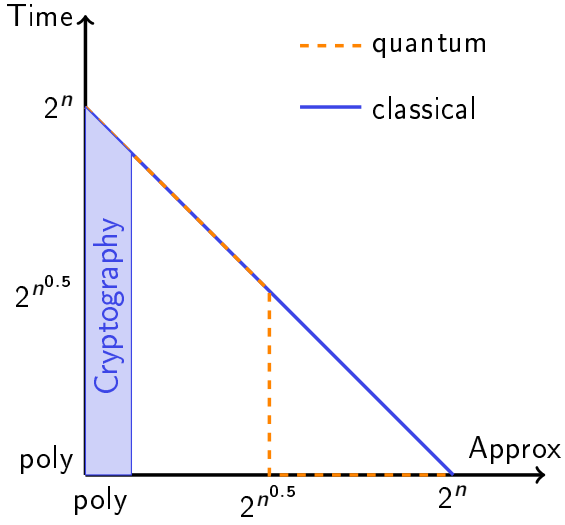
Hardness of SVP

Asymptotics:



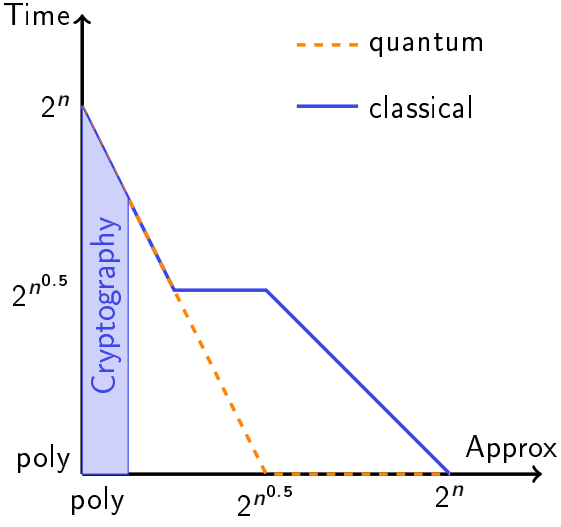
SVP and mod-SVP_k

($k \geq 2$)



id-SVP [CDW17]

(in cyclotomic fields)



id-SVP [PHS19,BR20]

(with $2^{O(n)}$ pre-processing)

[CDW17] Cramer, Ducas, Wesolowski. Short stickelberger class relations and application to ideal-SVP. Eurocrypt.

[PHS19] Pellet-Mary, Hanrot, Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

[BR20] Bernard, Roux-Langlois. Twisted-PHS: using the product formula to solve approx-SVP in ideal lattices. AC.

Hardness of SVP

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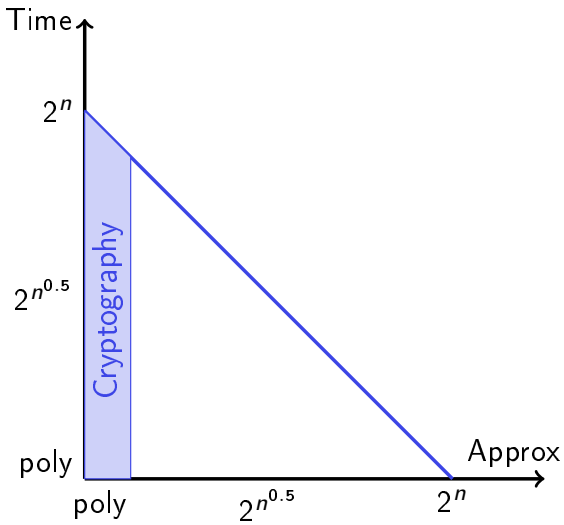

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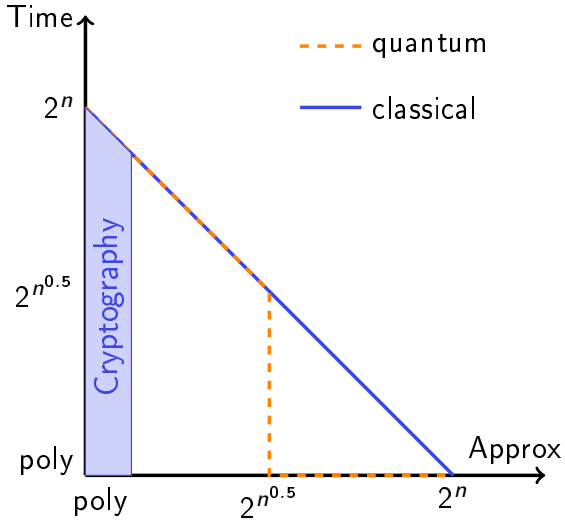
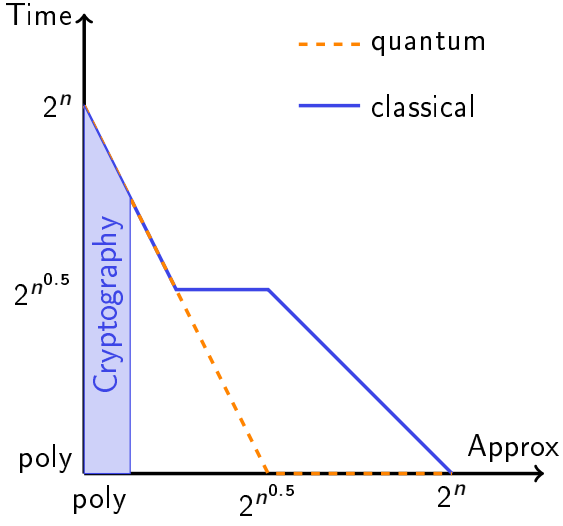


id-SVP [CDW17]

(in cyclotomic fields)



id-SVP [PHS19,BR20]

(with $2^{O(n)}$ pre-processing)

Practice: Darmstadt challenge[1]

$\rightsquigarrow$ max dim for SVP: 186

$\rightsquigarrow$ max dim for id-SVP: 176

[1] https://www.latticechallenge.org/

Ring and Module-LWE

(search) mod-LWE$_k$

Parameters: q and B

Problem: Sample

- $A \leftarrow \text{Uniform}((\mathcal{O}_K/(q\mathcal{O}_K))^{m\times k})$
- $s, e \in {\mathcal{O}_K}^k \times {\mathcal{O}_K}^m$ with coefficients in $\{-B, \cdots, B\}$

Given A and $b = A \cdot s + e \bmod q$, recover s

(size of s and e can be defined using Σ or σ)

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Module-LWE is a lattice problem:

mod-LWE$_k \approx$ CVP in a module lattice of rank m $(\Lambda_q^{\text{mod}}(A))$

NTRU [HPS98]

Parameters:
- q prime and large
- $B \ll \sqrt{q}$ integer

NTRU instance:
- sample f, g randomly in $\mathcal{O}_K$ with coefficients $\leq B$
- return $h = f \cdot g^{-1} \bmod q$

NTRU assumption: given h, there is no efficient algorithm that can find u and v such that $|u|, |v| \leq B$ and $h = u \cdot v^{-1} \bmod q$

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Now the NTRU assumption should hold

(or if it does not, then you have a paper!)

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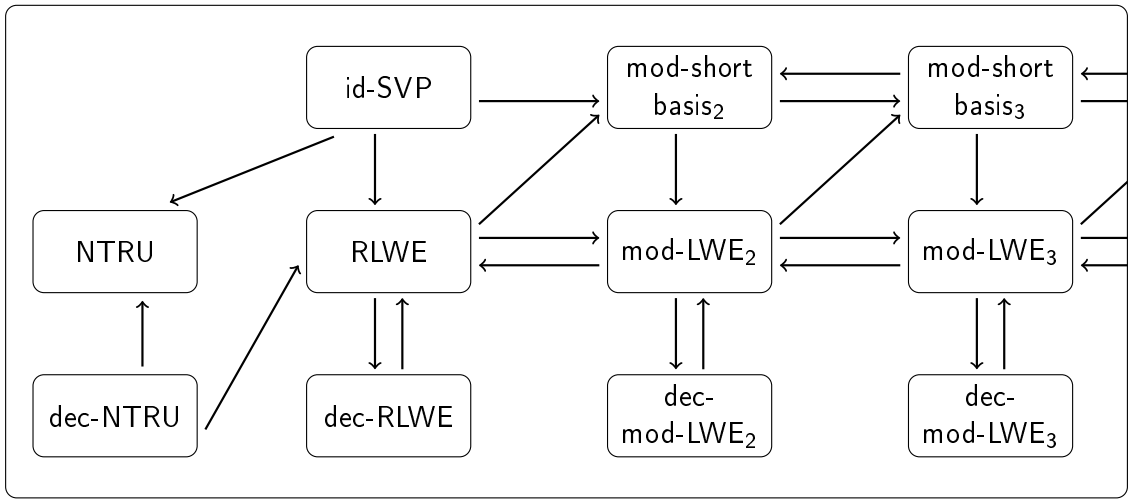
Reductions between problems

(or how to compare hardness of different problems)



LWE is as hard as worst-case lattice problems

Theorem [Reg05]

For any $m = \mathrm{poly}(n)$, modulus $q \leq 2^{\mathrm{poly}(n)}$ and $B \geq 2\sqrt{n}$, solving LWE is at least as hard as quantumly solving γ-SIVP on arbitrary n-dimensional lattice, for some approximation factor $\gamma = \tilde{O}(n \cdot q/B)$.

⚠ the reduction is for a variant of LWE where s and e are sampled from a discrete Gaussian distribution of parameter B ⚠

[Reg05] Regev. On lattices, learning with errors, random linear codes, and cryptography. STOC.

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[Pei09] Peikert. Public-key cryptosystems from the worst-case shortest vector problem. STOC.

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Solving LWE on average $\gtrsim$ (quantumly!) Computing short basis in any lattice of rank n

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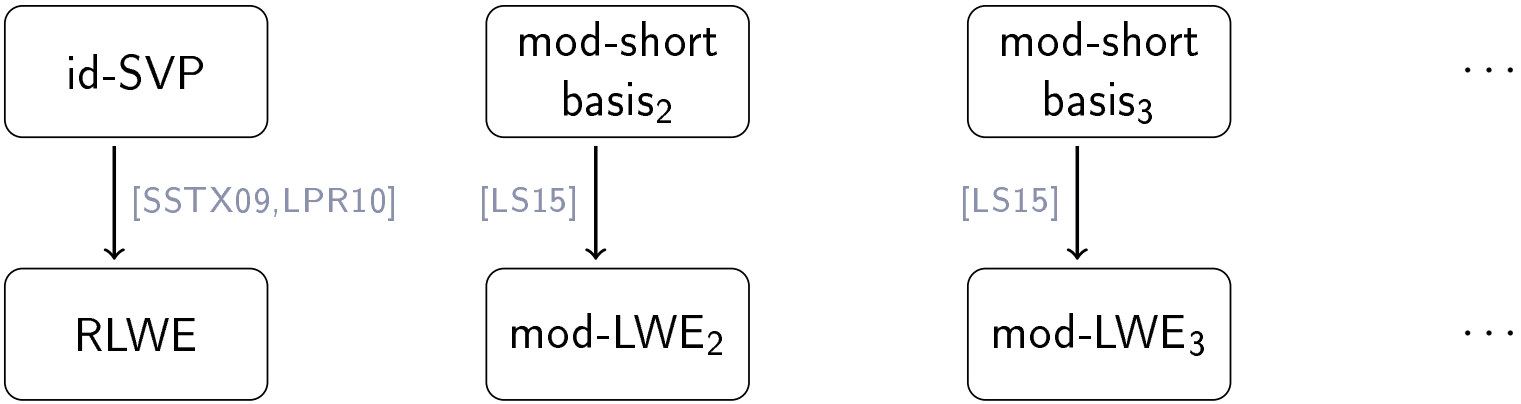

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- reductions may be quantum
- reductions hold for σ and Gaussian noise

[SSTX09] Stehlé, Steinfeld, Tanaka, Xagawa. Efficient public key encryption based on ideal lattices. Asiacrypt.

[LPR10] Lyubashevsky, Peikert, Regev. On ideal lattices and learning with errors over rings. Eurocrypt.

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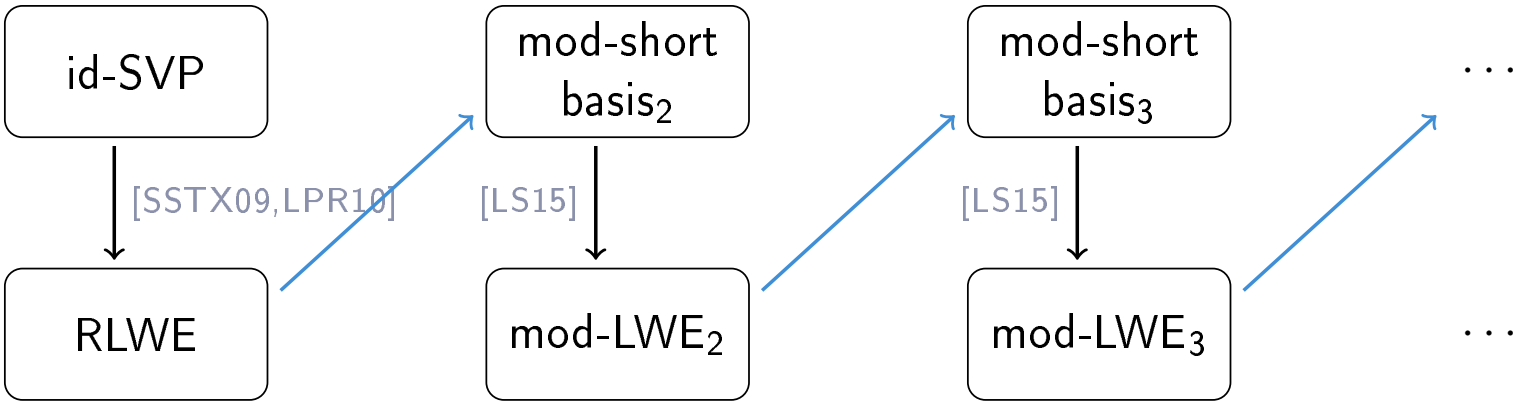

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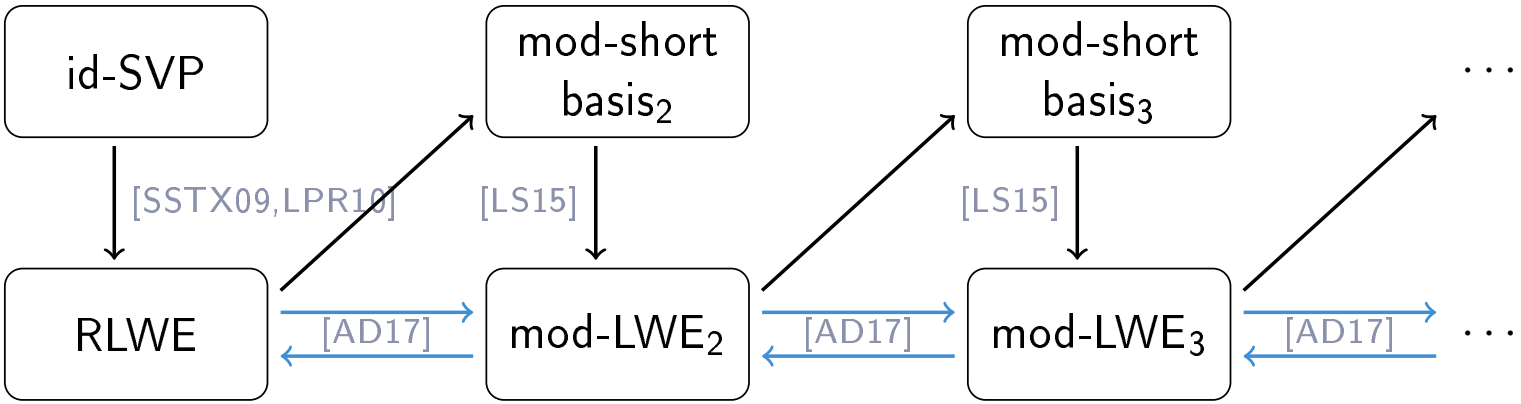

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[AD17] Albrecht, Deo. Large modulus ring-LWE $\geq$ module-LWE. Asiacrypt.

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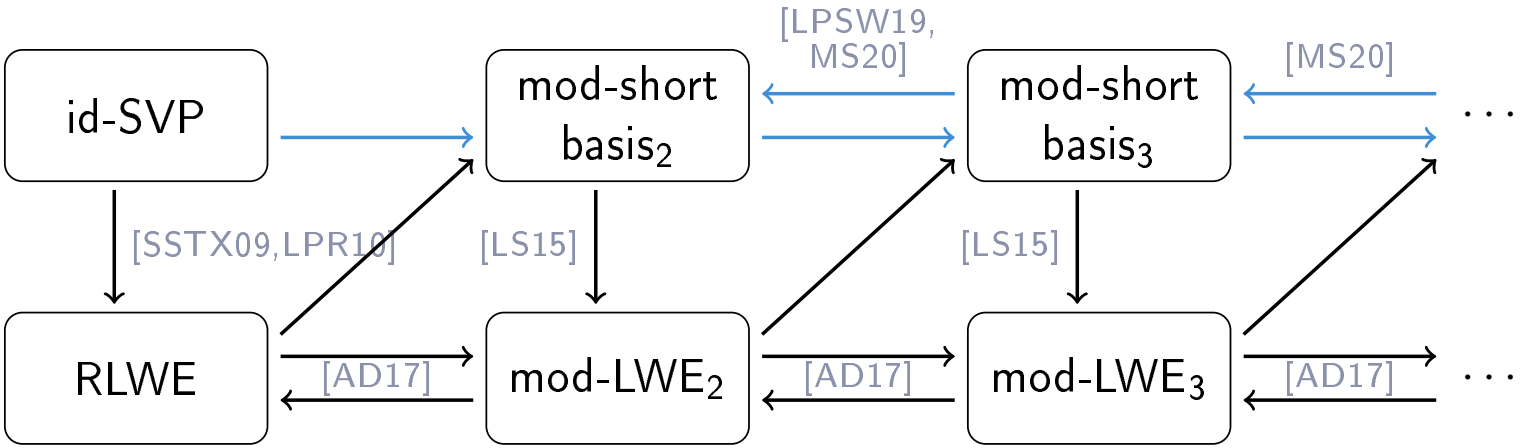

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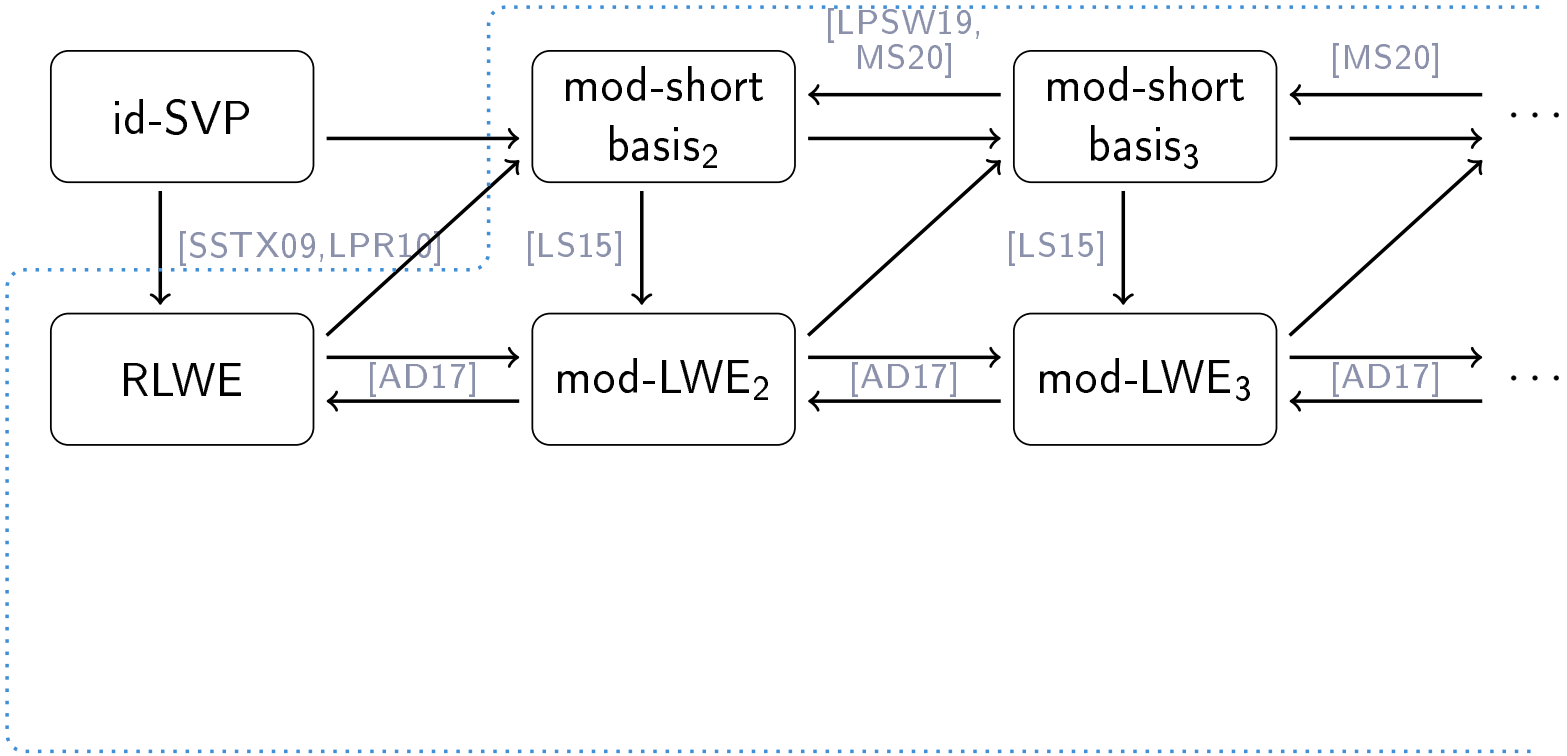

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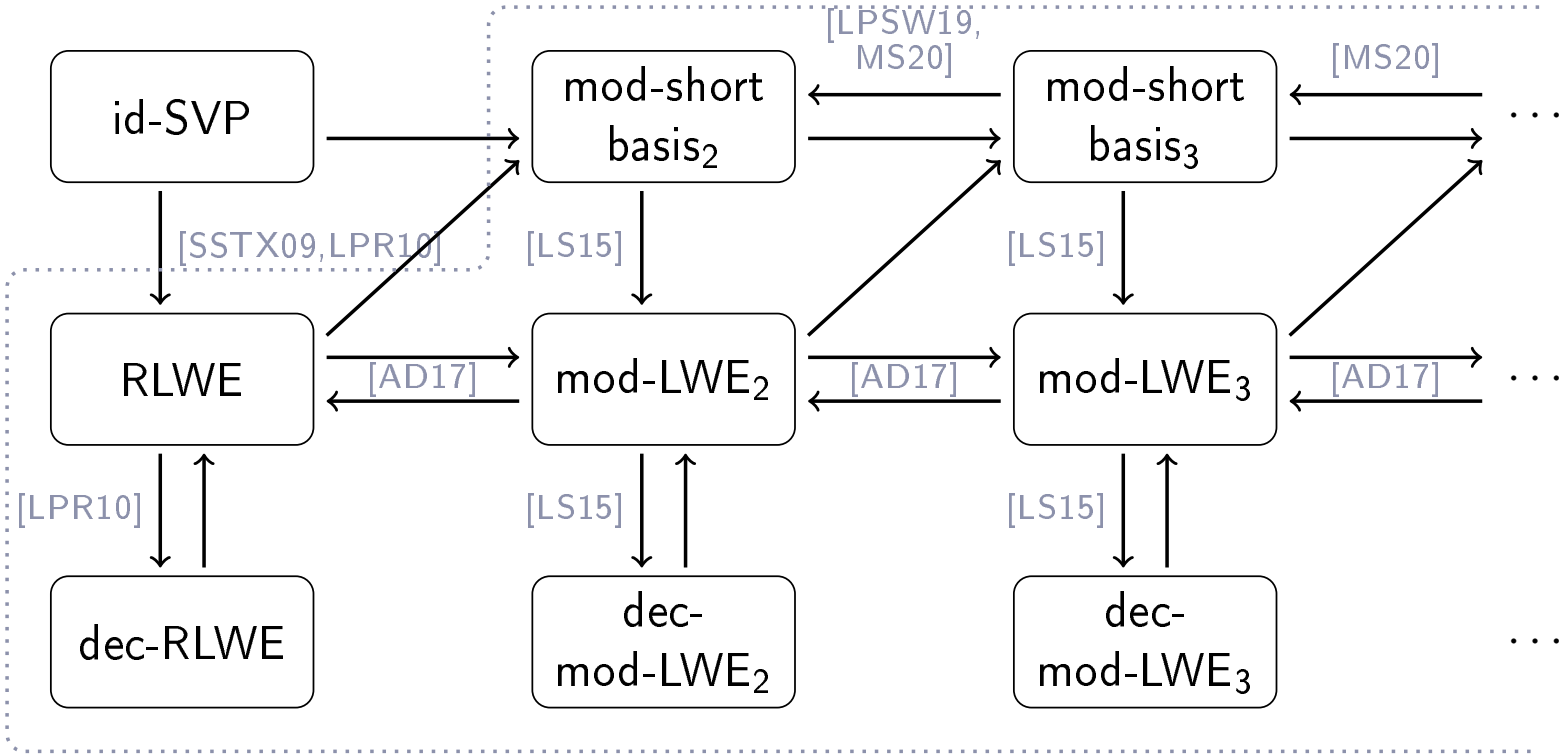

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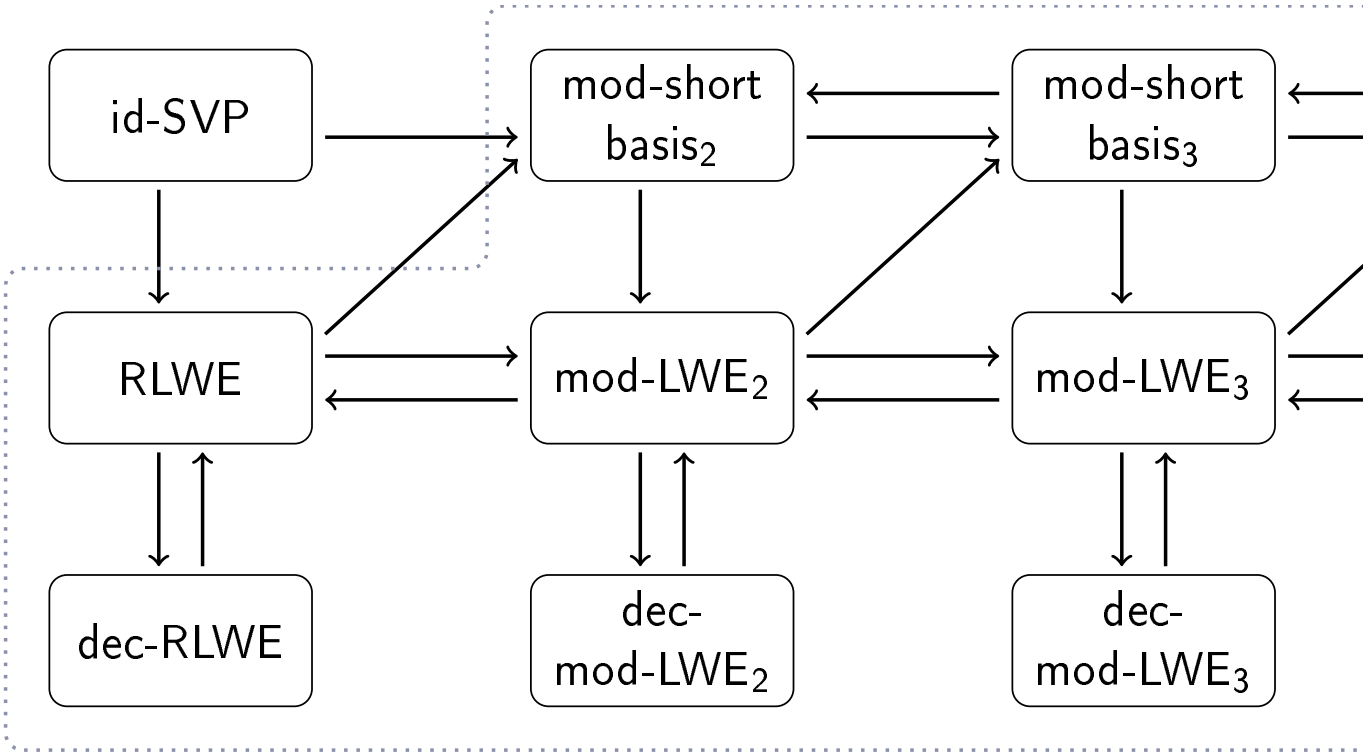

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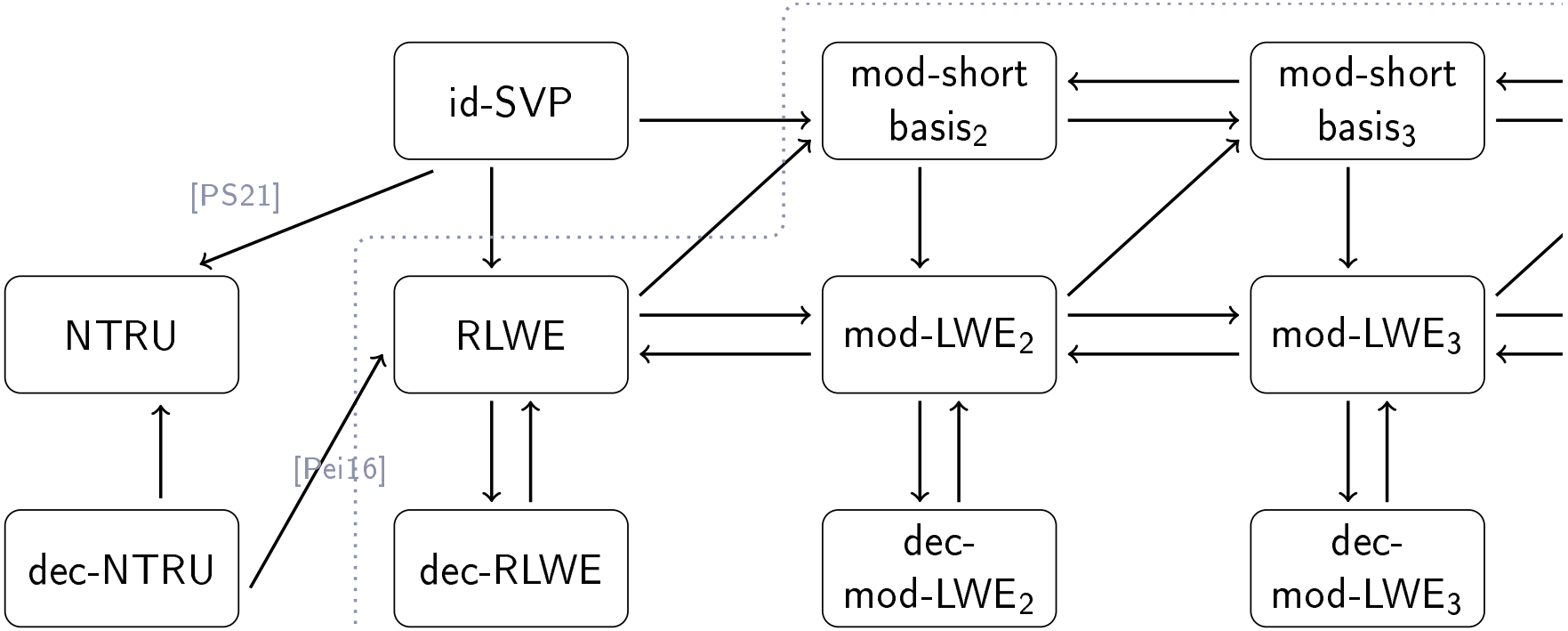

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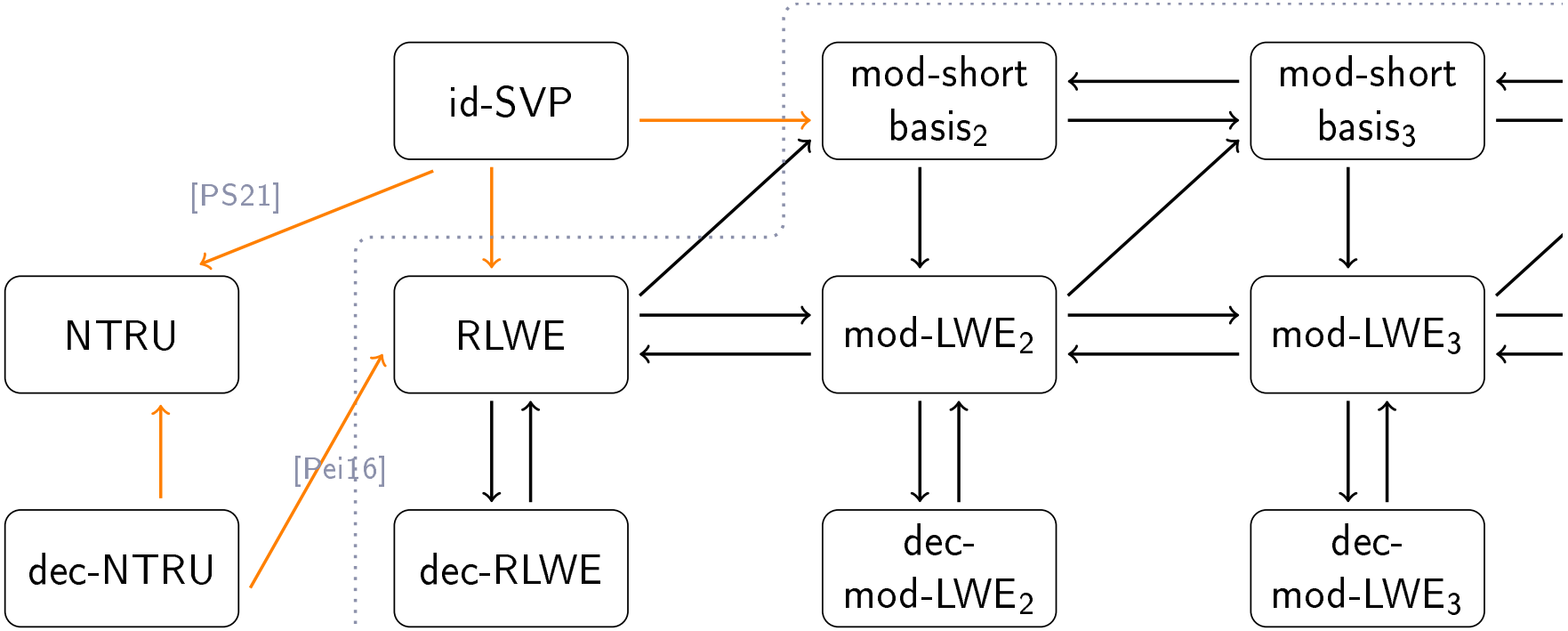

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Breaking id-SVP do break:

- some early FHE schemes
- the PV-Knap problem [HPS+14,BSS22]

[HPS+14] Hoffstein, Pipher, Schanck, Silverman, and Whyte. Practical signatures from the partial Fourier recovery problem. ACNS.

[BSS22] Boudgoust, Sakzad, and Steinfeld. Vandermonde meets Regev: Public Key Encryption Schemes Based on Partial Vandermonde Problems. DCC.

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