Introduction to lattice-based cryptography

Alice Pellet-Mary

Amusec 2024, Marseille

Context: example

Diffie-Hellman key exchange

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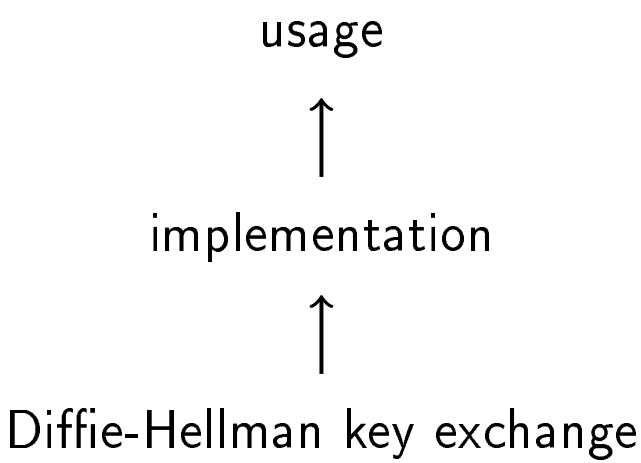

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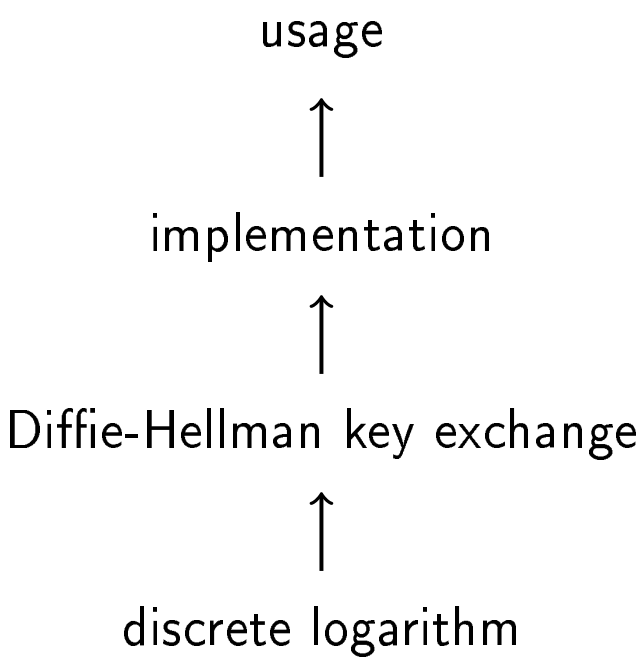

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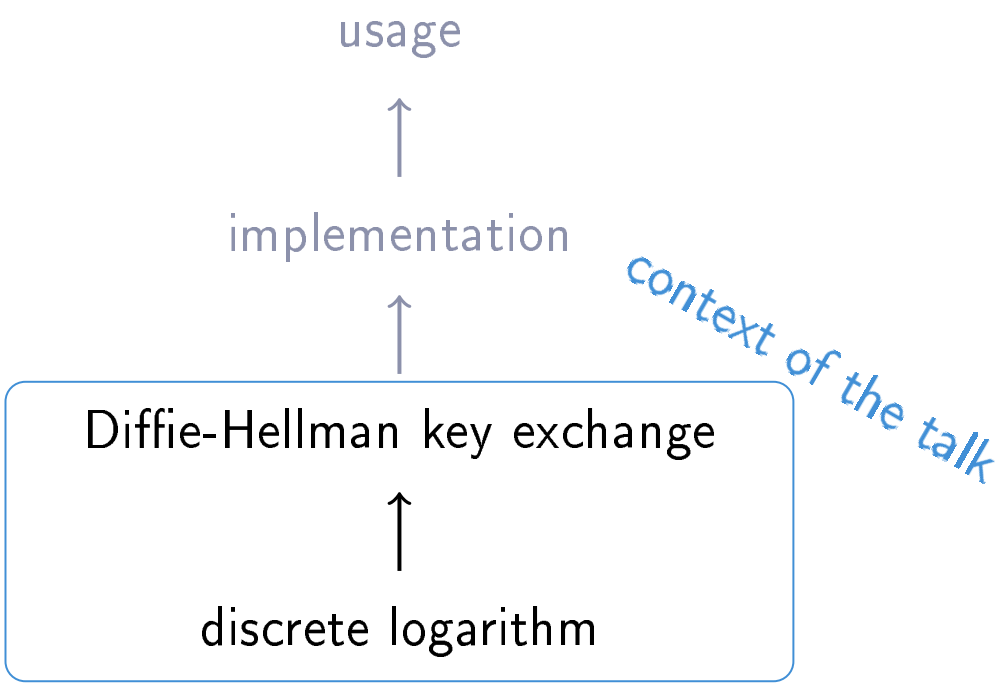

Zooming in: public key cryptography

Cryptographic primitives

public key encryption | signature | homomorphic encryption | ...

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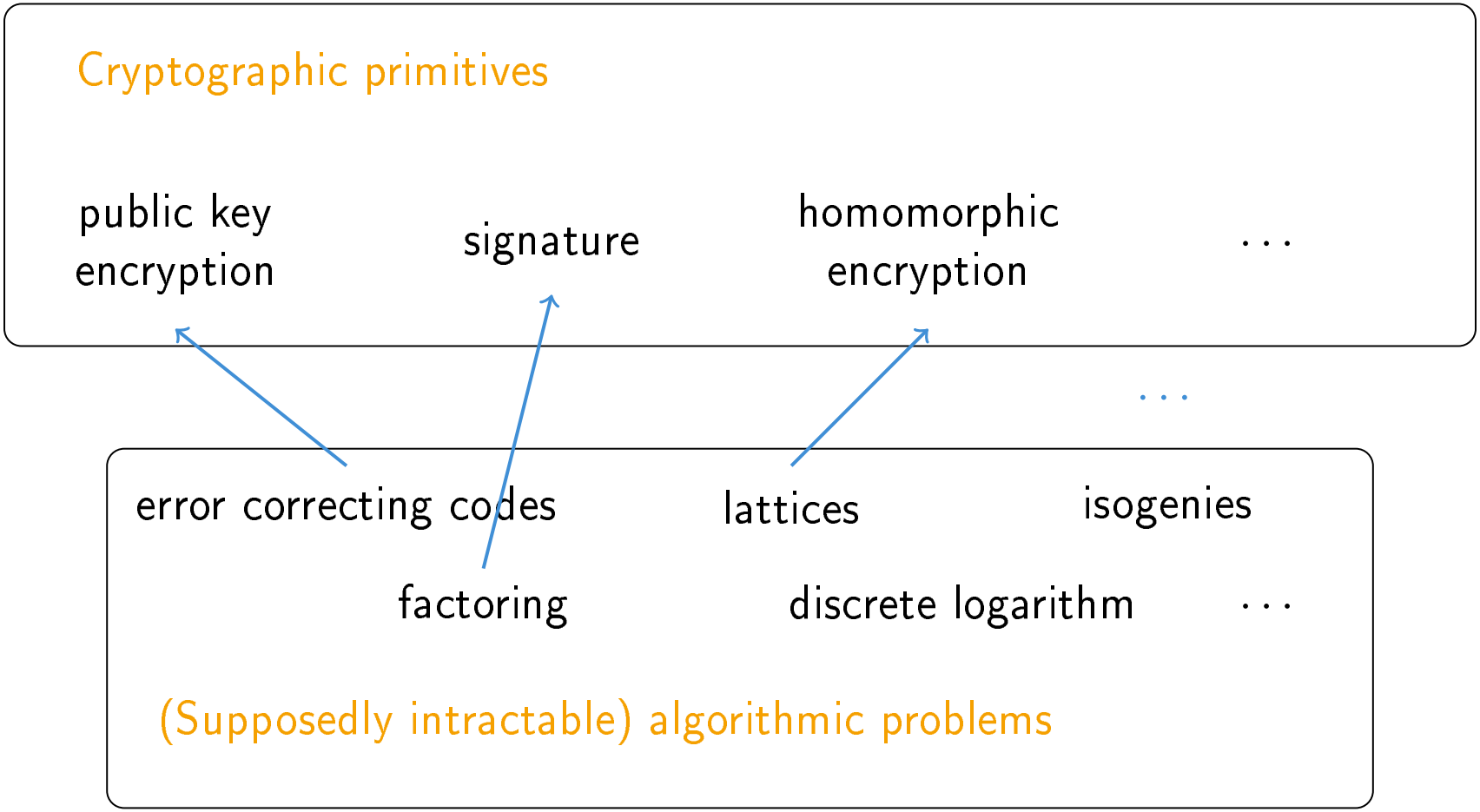
error correcting codes | lattices | isogenies | factoring | discrete logarithm | ...

(Supposedly intractable) algorithmic problems

Zooming in: public key cryptography



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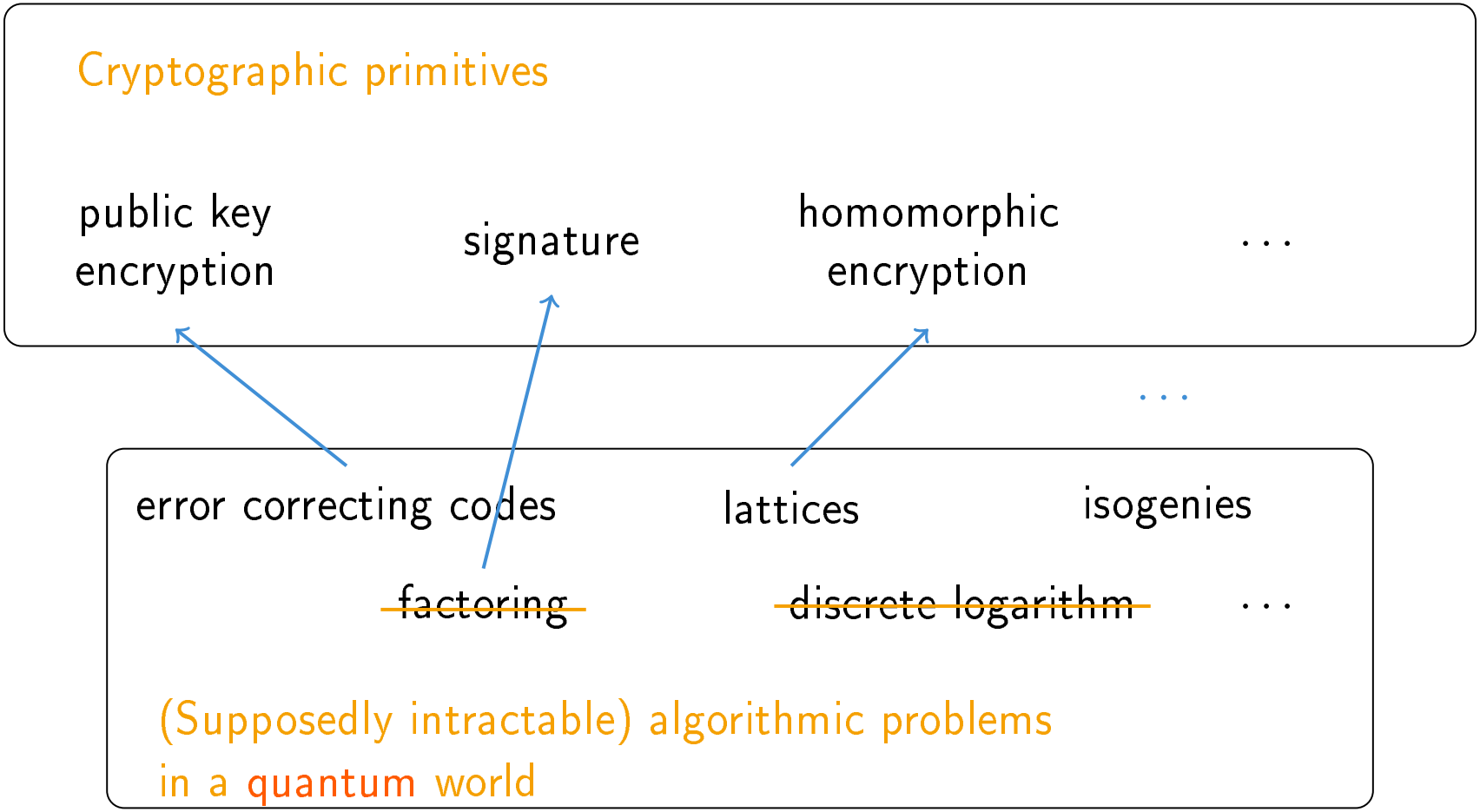

"quantum world" = assuming attackers have access to a quantum computers

Post-quantum cryptography vs quantum cryptography

Post-quantum cryptography:

- uses classical infrastructures
- resists to attackers with quantum computers

Quantum cryptography:

- uses quantum mechanics and dedicated infrastructures
- resists to attackers with quantum computers

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$\Rightarrow$ ANSSI recommends post-quantum cryptography[1]

ANSSI: Agence nationale de la sécurité des systèmes d'information (France)

[1] https://cyber.gouv.fr/en/publications/should-quantum-key-distribution-be-used-secure-communications

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NIST and ANSSI decided that the threat cannot be neglected

- even if the probability is very small, do we want to risk it?
- changing all cryptographic protocols takes time

NIST: the National Institute of Standards and Technology (USA)

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- even if the probability is very small, do we want to risk it?
- changing all cryptographic protocols takes time

$\Rightarrow$ they encourage a progressive transition to post-quantum cryptography

Post-quantum cryptography is going to be deployed
in a (relatively) near future

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NIST standardization process

NIST competition for post-quantum key exchange and signatures:

https://csrc.nist.gov/projects/post-quantum-cryptography

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  (some others are still being analyzed)
  - 1 key exchange: Kyber (lattices)
  - 3 signatures: Dilithium (lattices), Falcon (lattices), SPHINCS+ (hash)

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  - 1 key exchange: Kyber (lattices)
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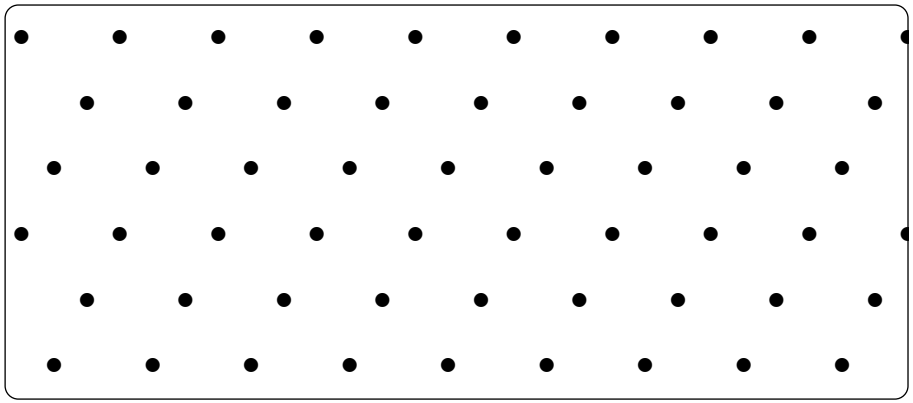
- 2023 call for additional digital signatures ("not lattice-based")
  - 40 submissions, still under consideration

https://csrc.nist.gov/projects/post-quantum-cryptography

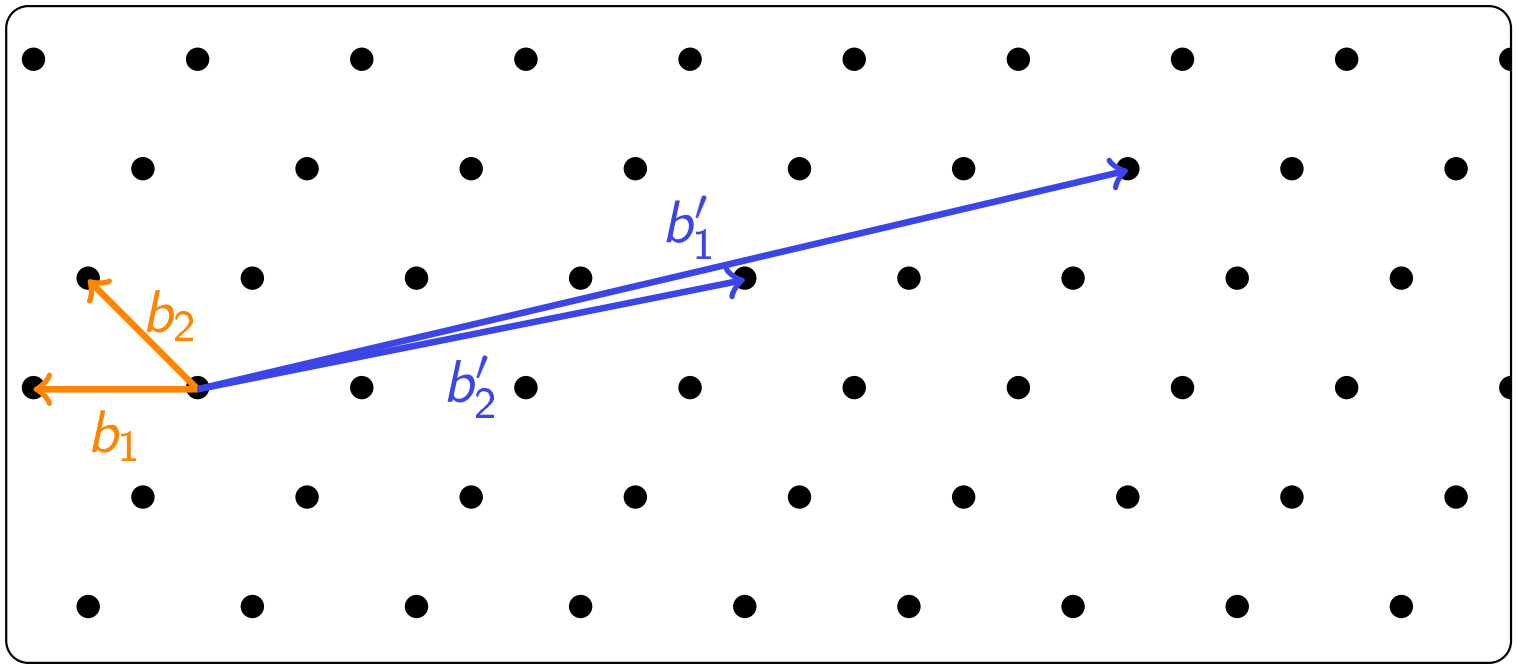
Plan of the talk

1. Lattices and lattice problems
2. Are lattice problems (quantumly) hard?
3. Constructing signatures from lattices
4. Falcon's signature scheme

Lattices and lattice problems



Lattices



- $\mathcal{L} = \{\sum_{i=1}^{n} x_i b_i \mid \forall i,\ x_i \in \mathbb{Z}\}$ is a lattice
- $(b_1, \ldots, b_n) =: B \in \mathrm{GL}_n(\mathbb{R})$ is a basis (not unique)
- n is the dimension (or rank)

Algorithmic problems on lattices

Example of problems:

(1) Testing equality of lattices

(2) Intersecting two lattices

(3) Computing a short basis of a lattice

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Quiz: which ones are easy or hard?

easy: polynomial time hard: no polynomial time algorithm known

Algorithmic problems on lattices

Example of problems:

(1) Testing equality of lattices $\Rightarrow$ easy

(2) Intersecting two lattices $\Rightarrow$ easy

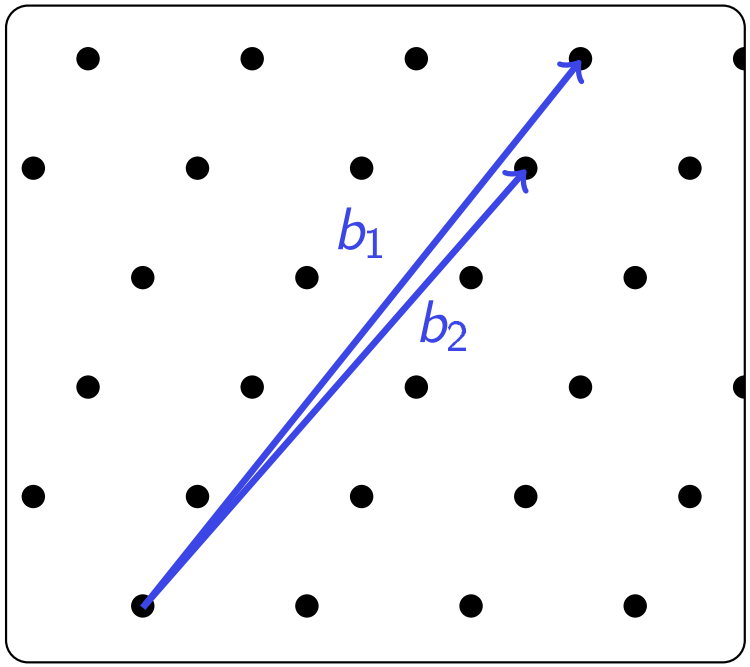
(3) Computing a short basis of a lattice $\Rightarrow$ hard

Quiz: which ones are easy or hard?

easy: polynomial time hard: no polynomial time algorithm known

Short basis problem

Input:



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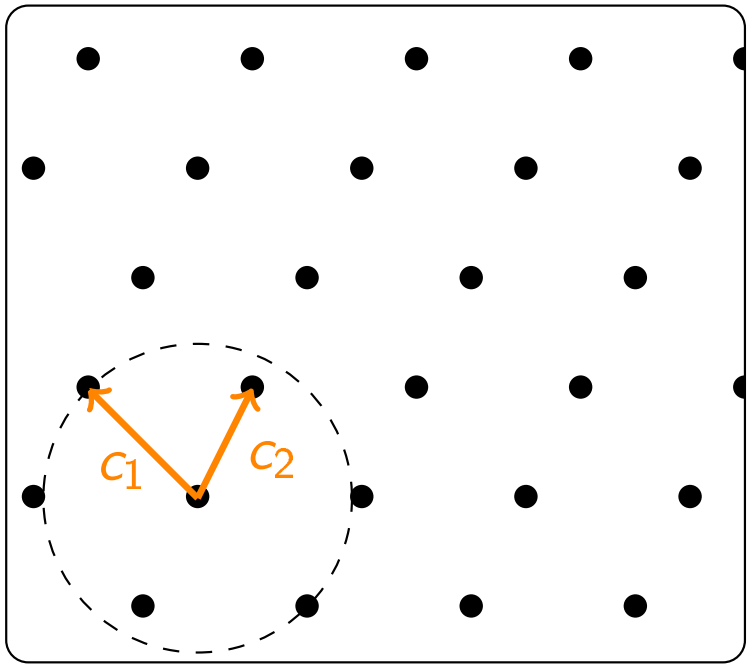

Shortest basis problem

$$\max_i \|c_i\| \leq \min_{B' \text{ basis of } \mathcal{L}} \left(\max_i \|b'_i\| \right)$$

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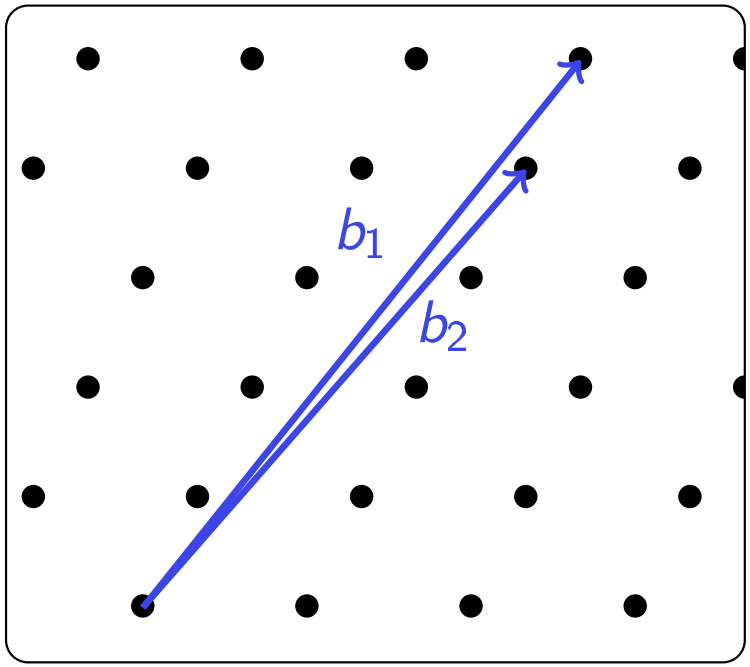

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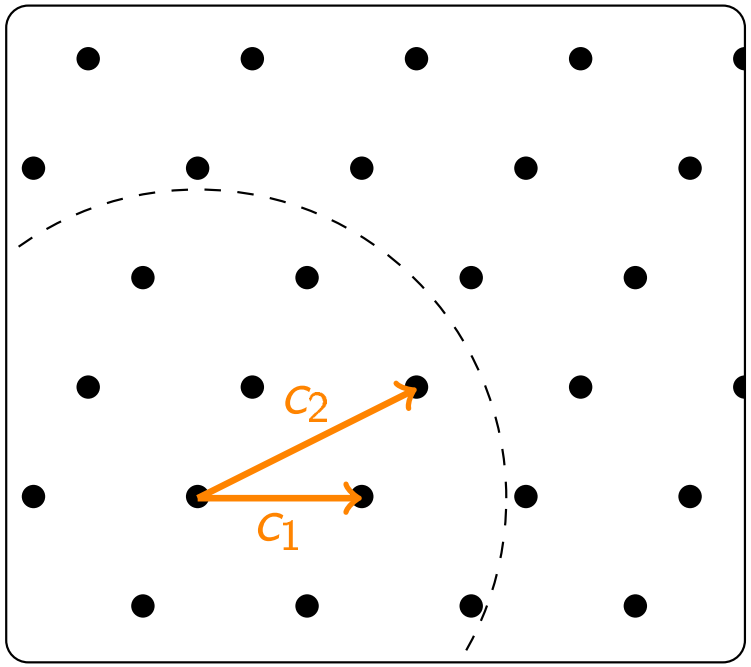


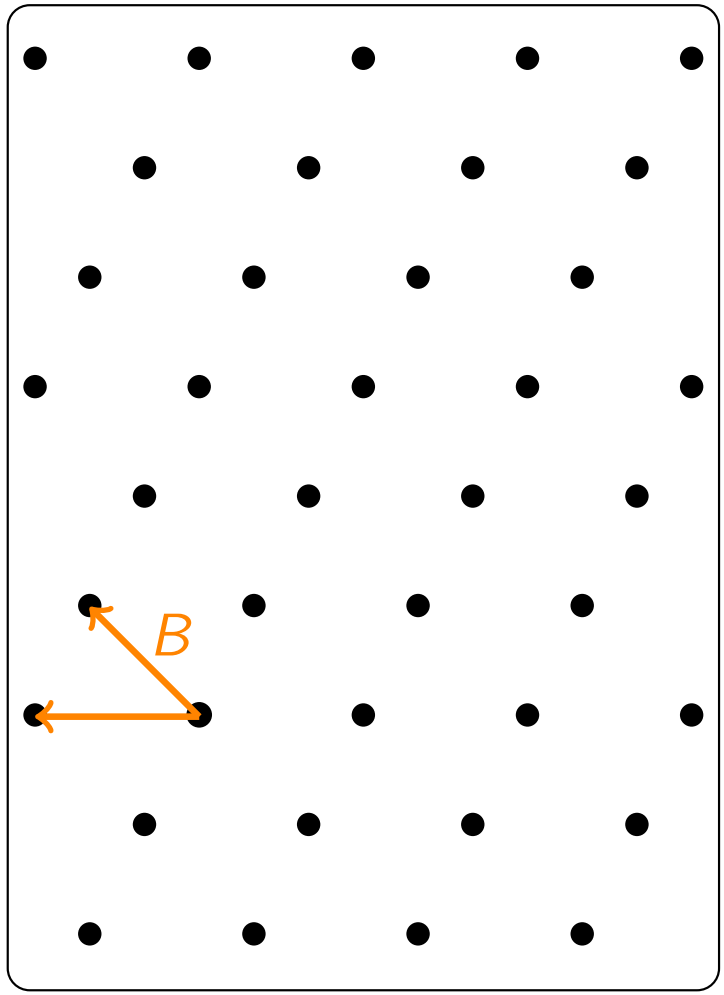
Approximate short basis problem

$$\max_i \|c_i\| \leq \gamma \cdot \min_{B' \text{ basis of } \mathcal{L}} \left(\max_i \|b'_i\| \right)$$

Digression: canonical representation

Representation of a lattice $\mathcal{L}$:

a basis $B \in \mathbb{Z}^{n \times n}$ of $\mathcal{L}$



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Difficulty:

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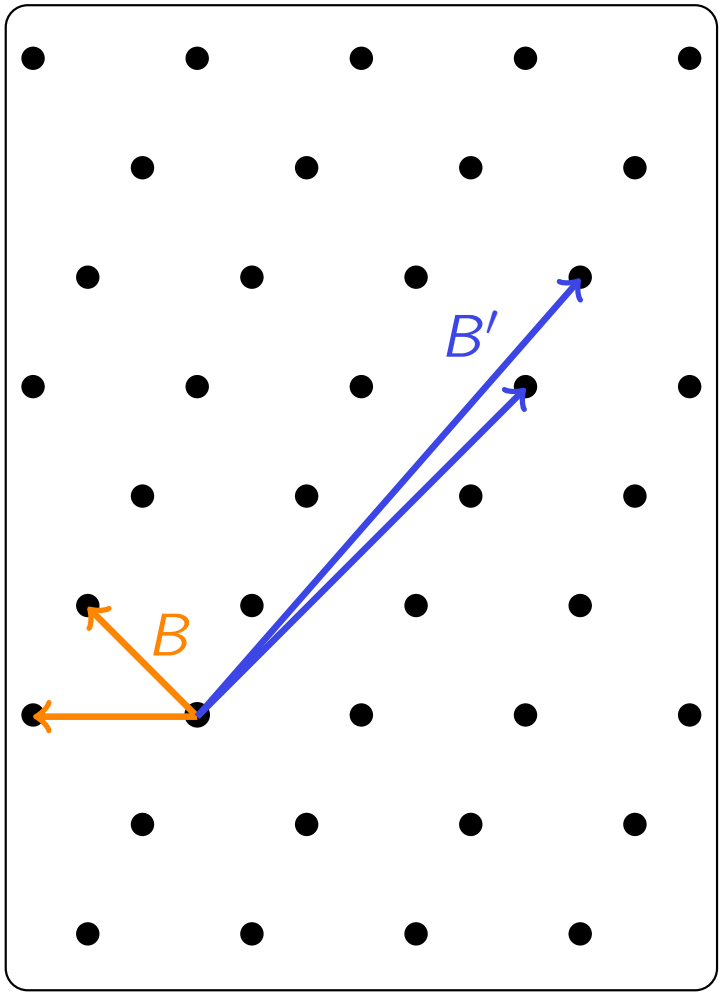

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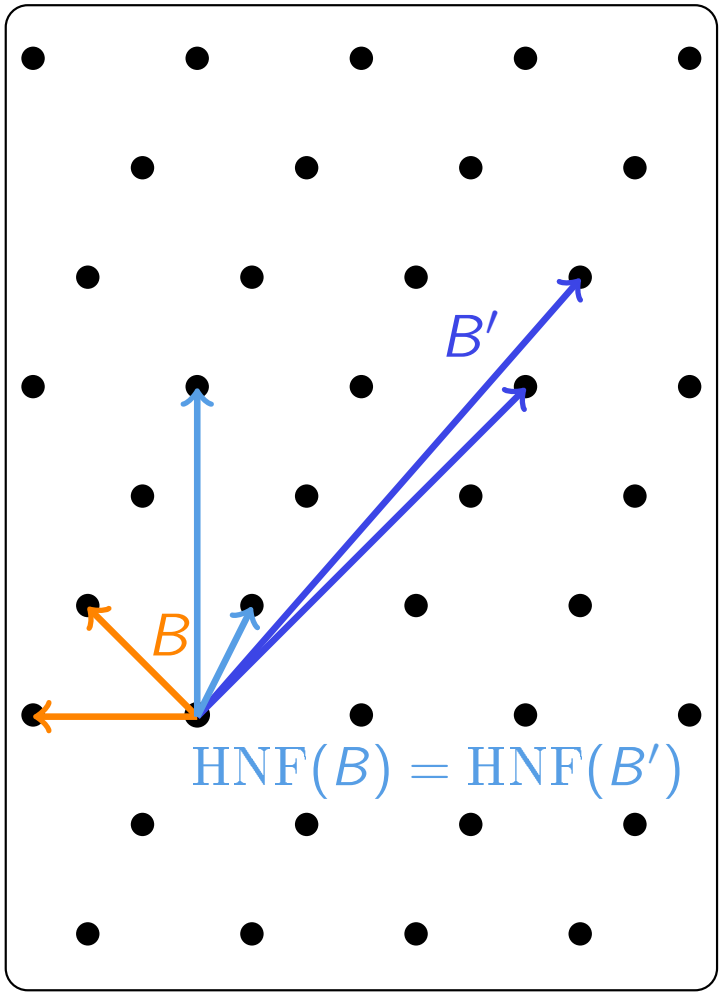
Solution: take the Hermite Normal Form (HNF) of any B

- it is unique $(\mathrm{HNF}(B) = \mathrm{HNF}(B'))$
- it is efficiently computable



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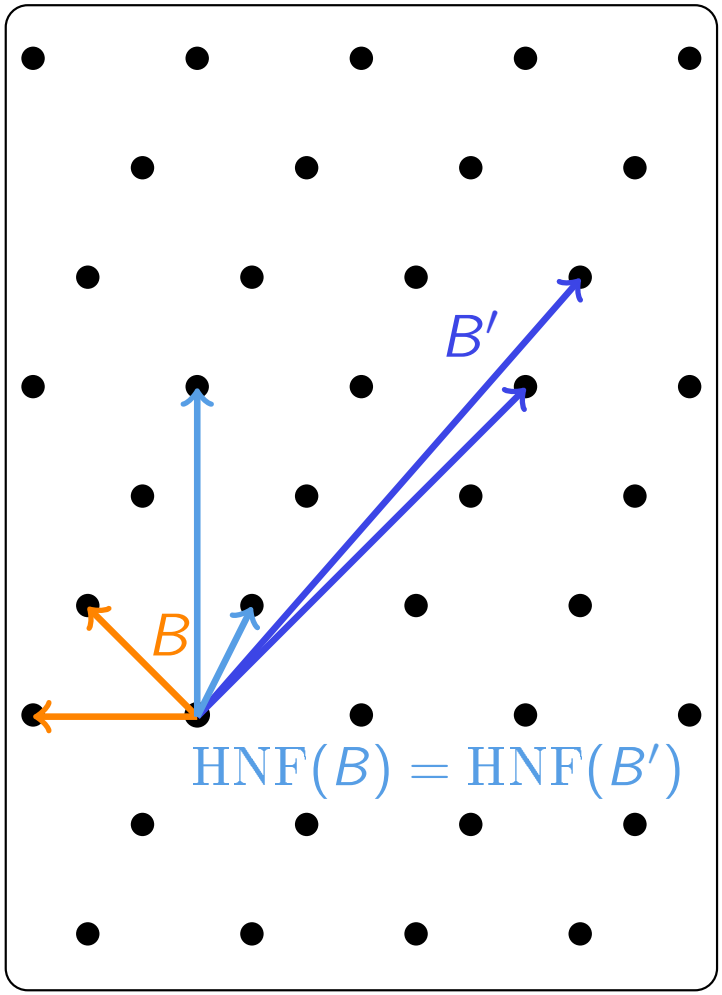
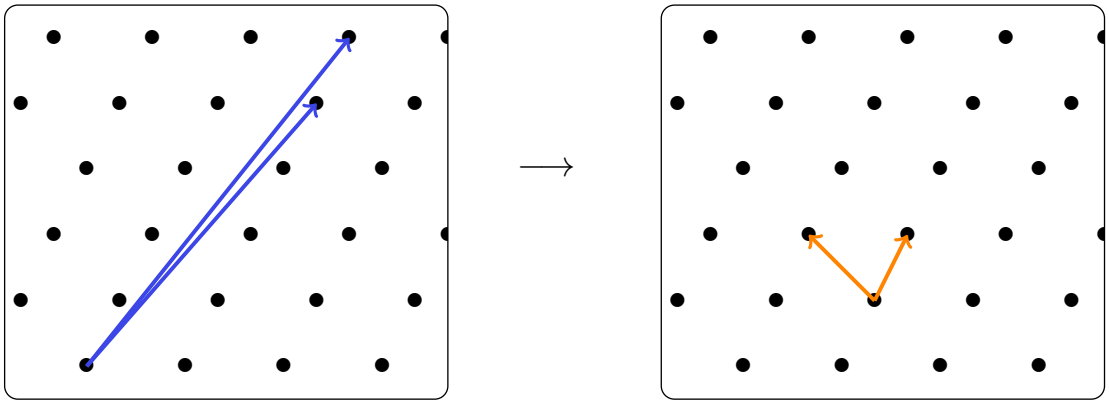
$\Rightarrow$ canonical representation of $\mathcal{L}$
(i.e., worse basis ever)



Section's conclusion

We have seen one lattice problem: the (approximate-)short basis problem



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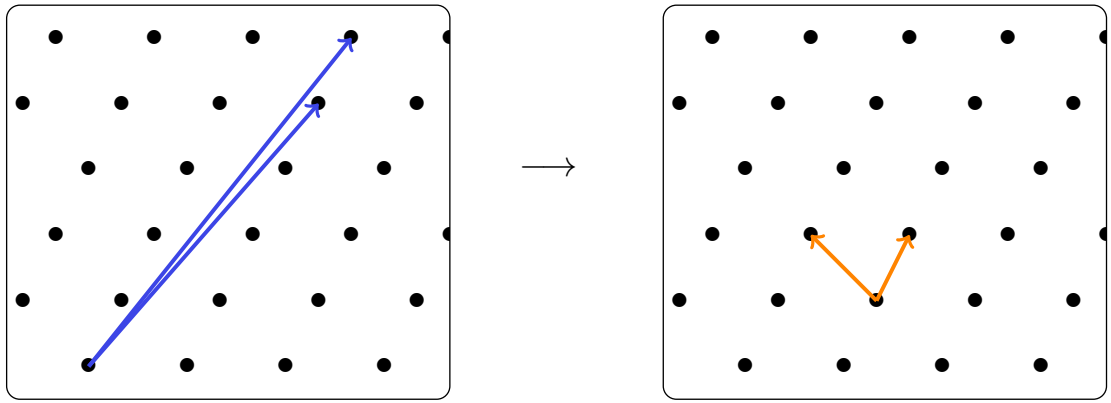


There exists many other lattice problems:

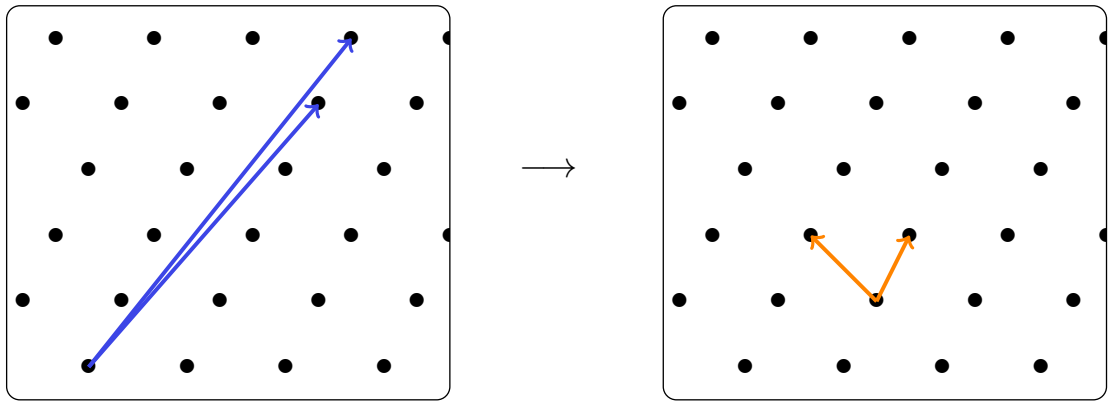
- the shortest vector problem
- the decoding problem (see later in the talk)
- ...

$\Rightarrow$ all of them are somehow equivalent in hardness

Nice survey of many lattice problems by Stephens-Davidowitz: http://www.noahsd.com/latticeproblems.pdf

Lattice reduction algorithms

or how to compute a short basis from a long one



Dimension 2: Lagrange-Gauss algorithm

video

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video

Theorem: The algorithm

- finds a shortest basis
- runs in polynomial time

The LLL algorithm [LLL82]

Input: basis $B = (b_1, \ldots, b_n)$

[LLL82] Lenstra, Lenstra, and Lovász. Factoring polynomials with rational coefficients. Mathematische annalen.

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Main idea: improve the basis locally on blocks of dimension 2

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- finds an approximate short basis with $\gamma = 2^n$

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  - run Lagrange-Gauss on L_i

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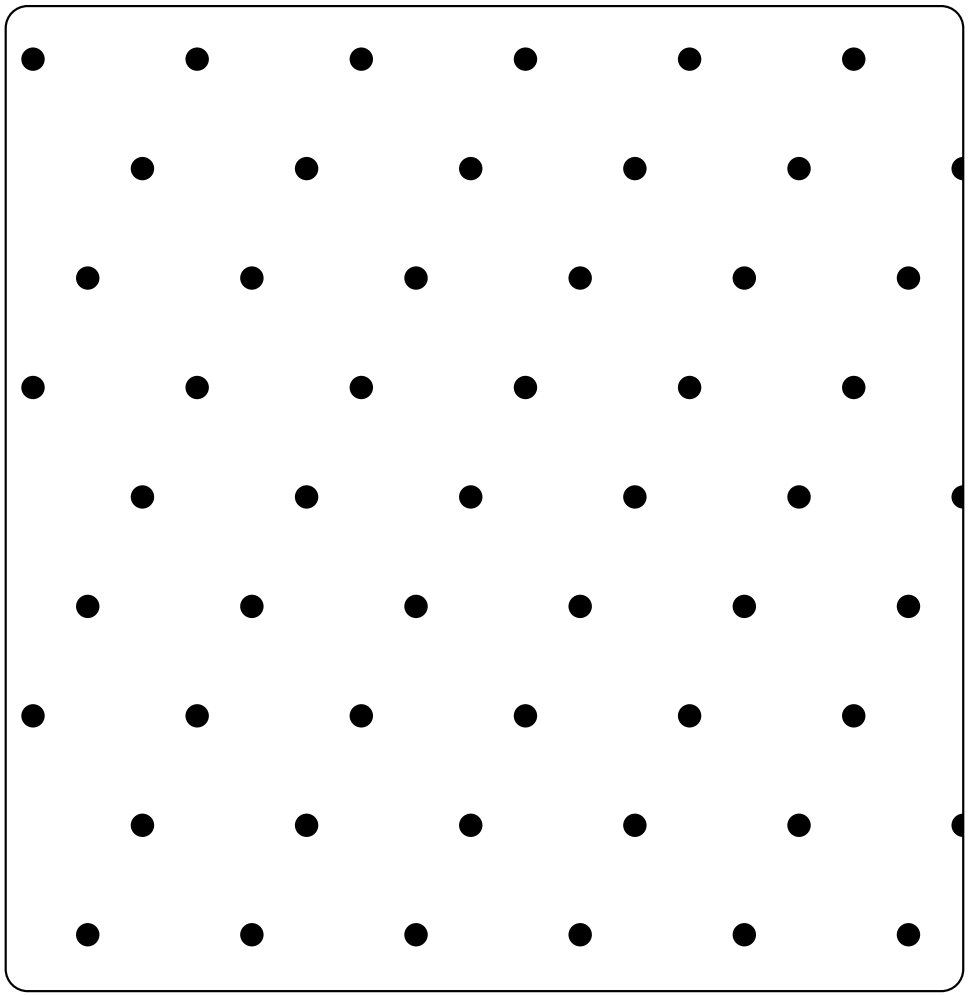
Sieving algorithm [AKS01]



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[AKS01] Ajtai, Kumar, and Sivakumar. A sieve algorithm for the shortest lattice vector problem. STOC

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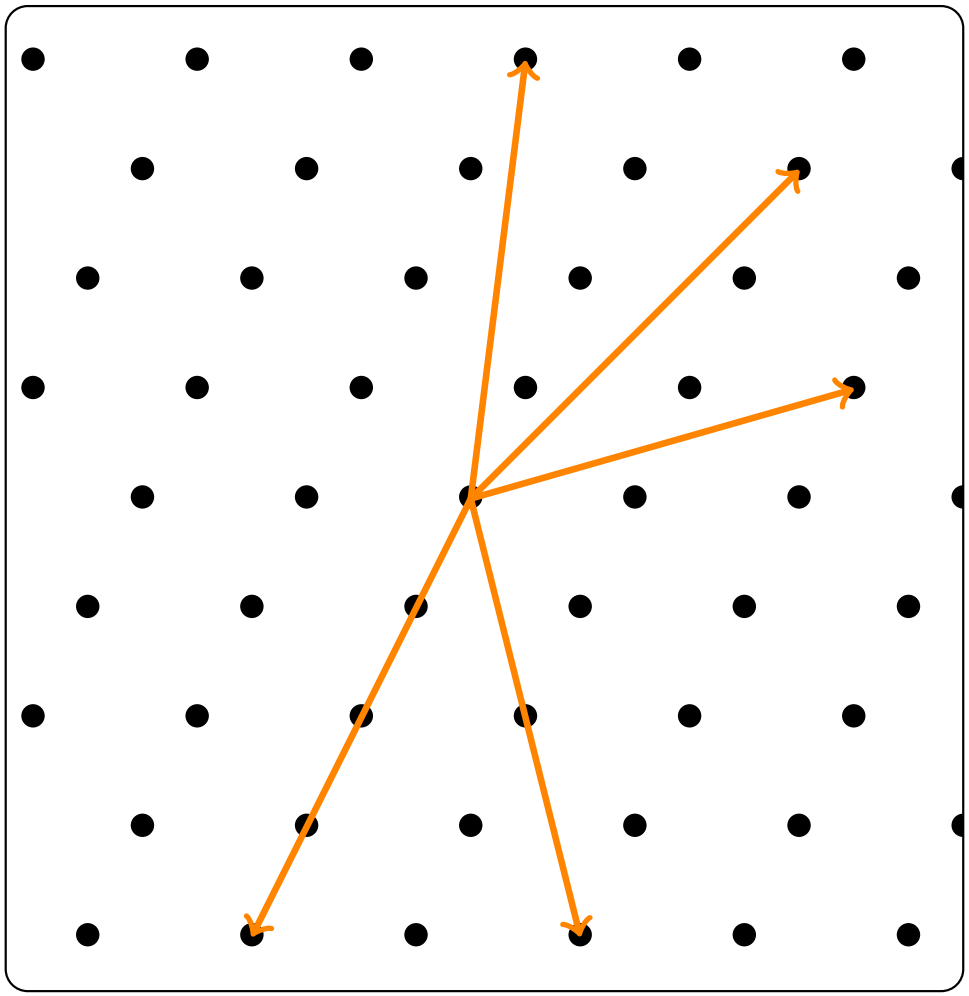

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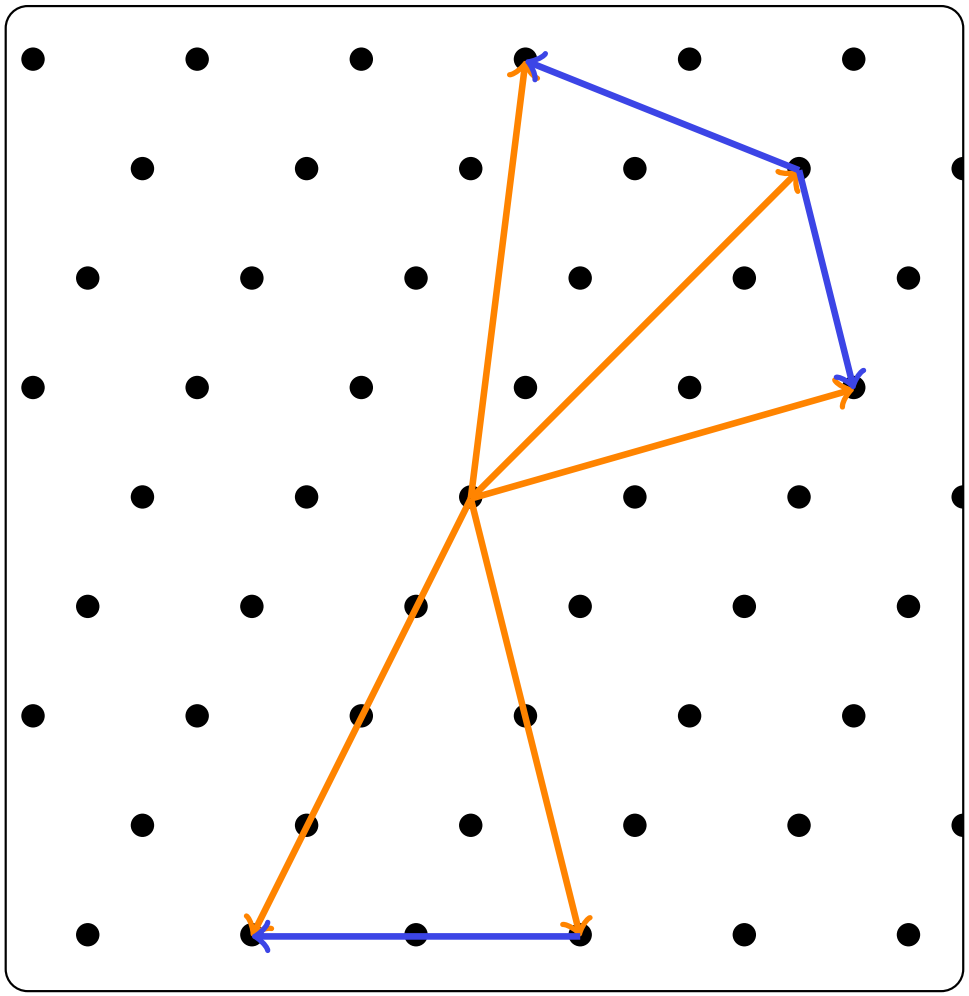

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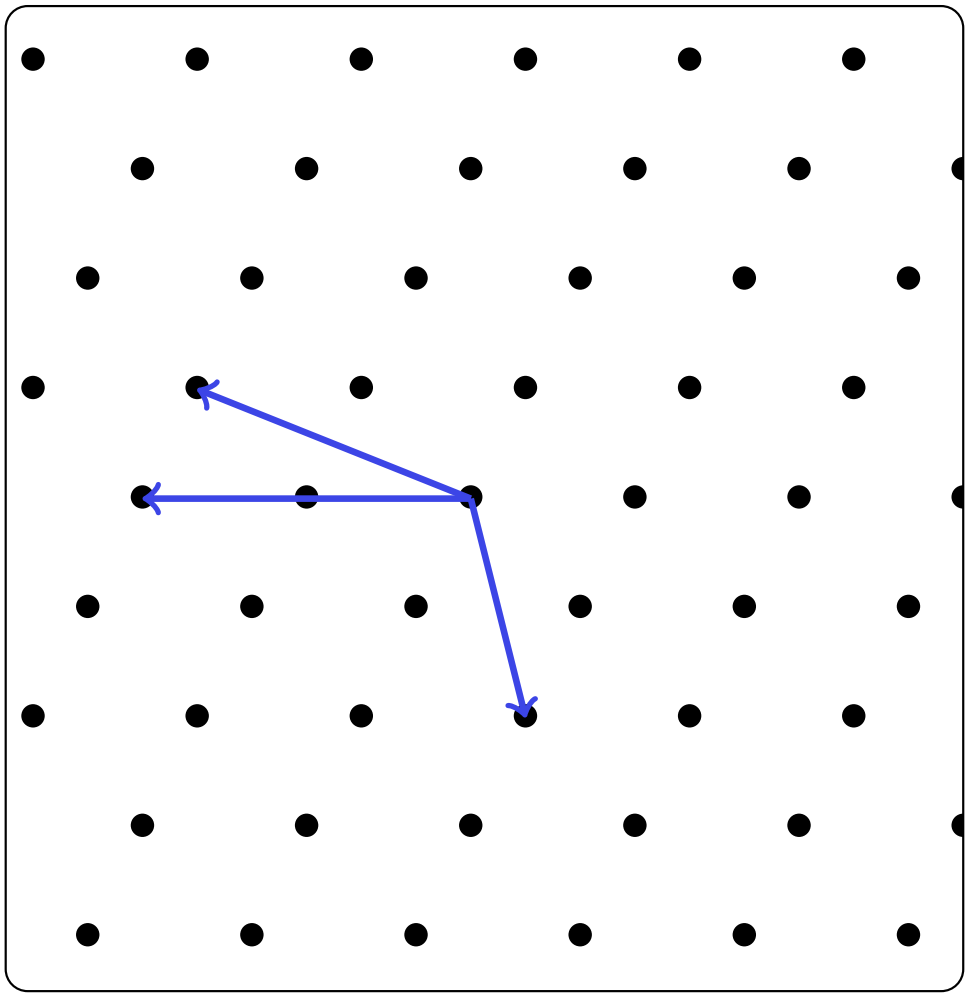

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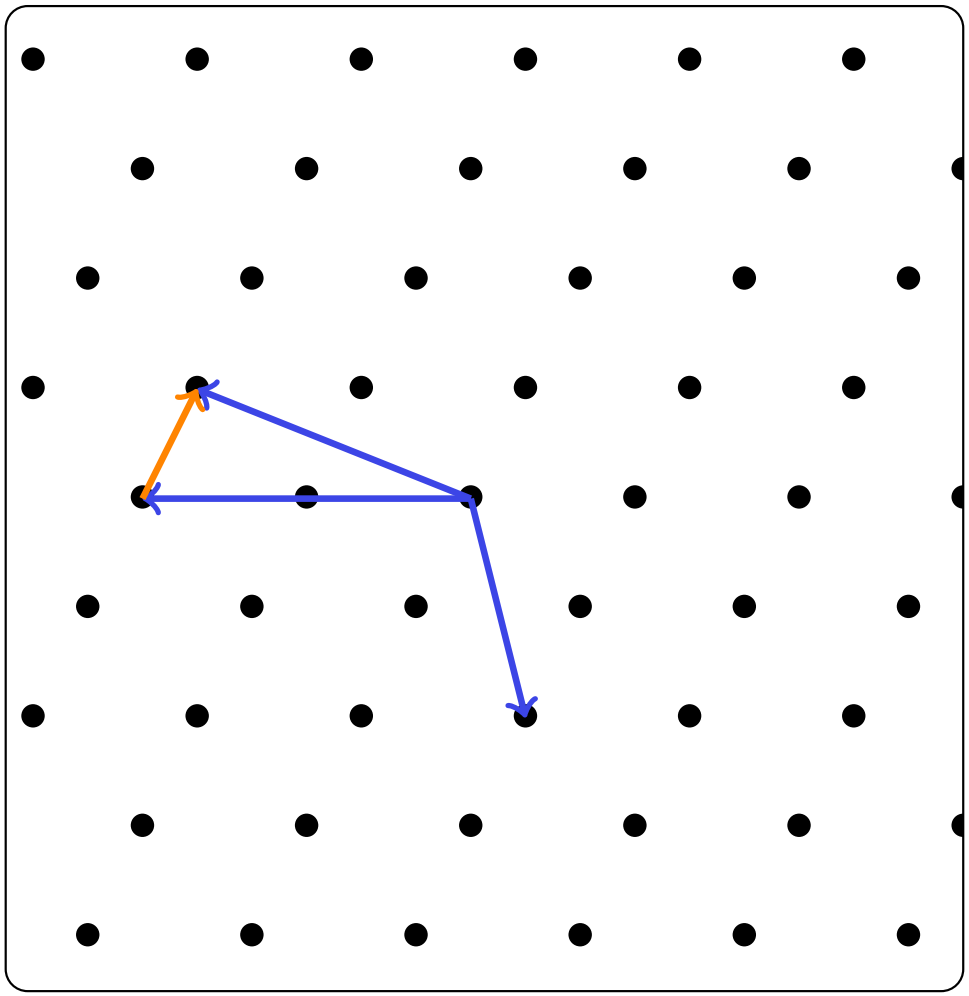

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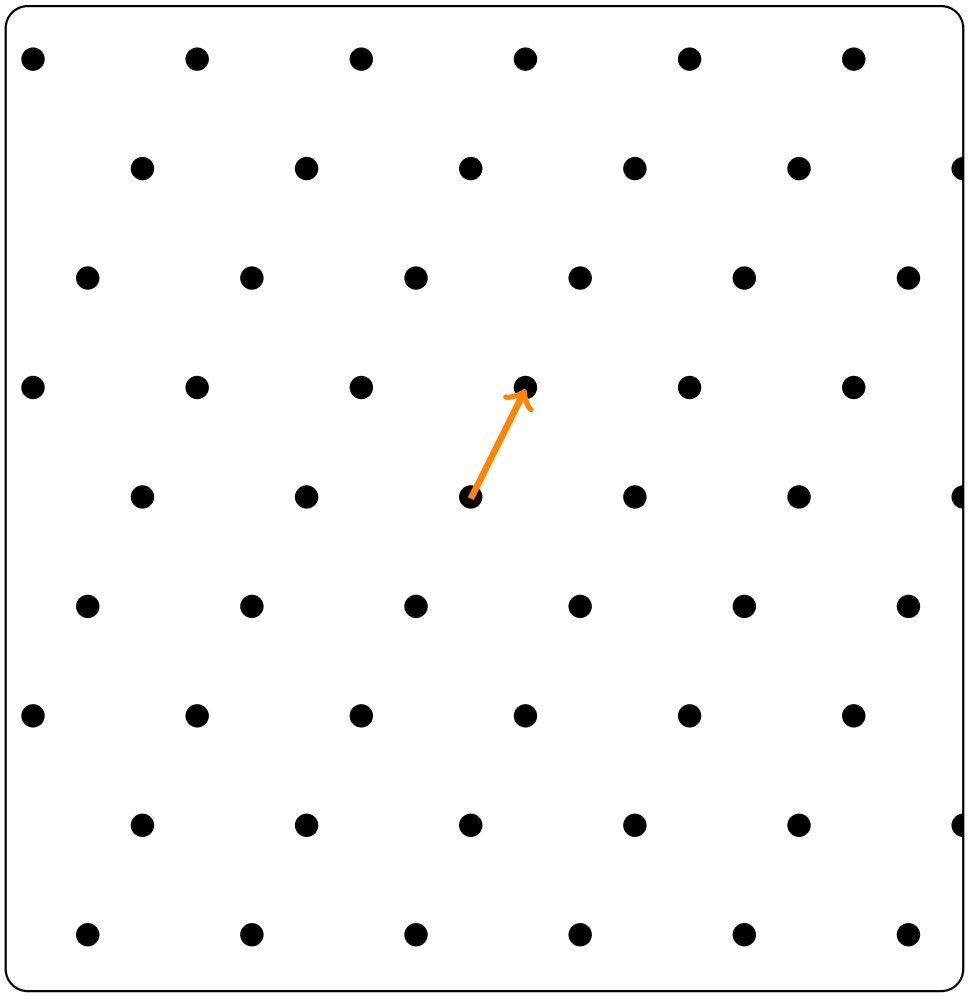

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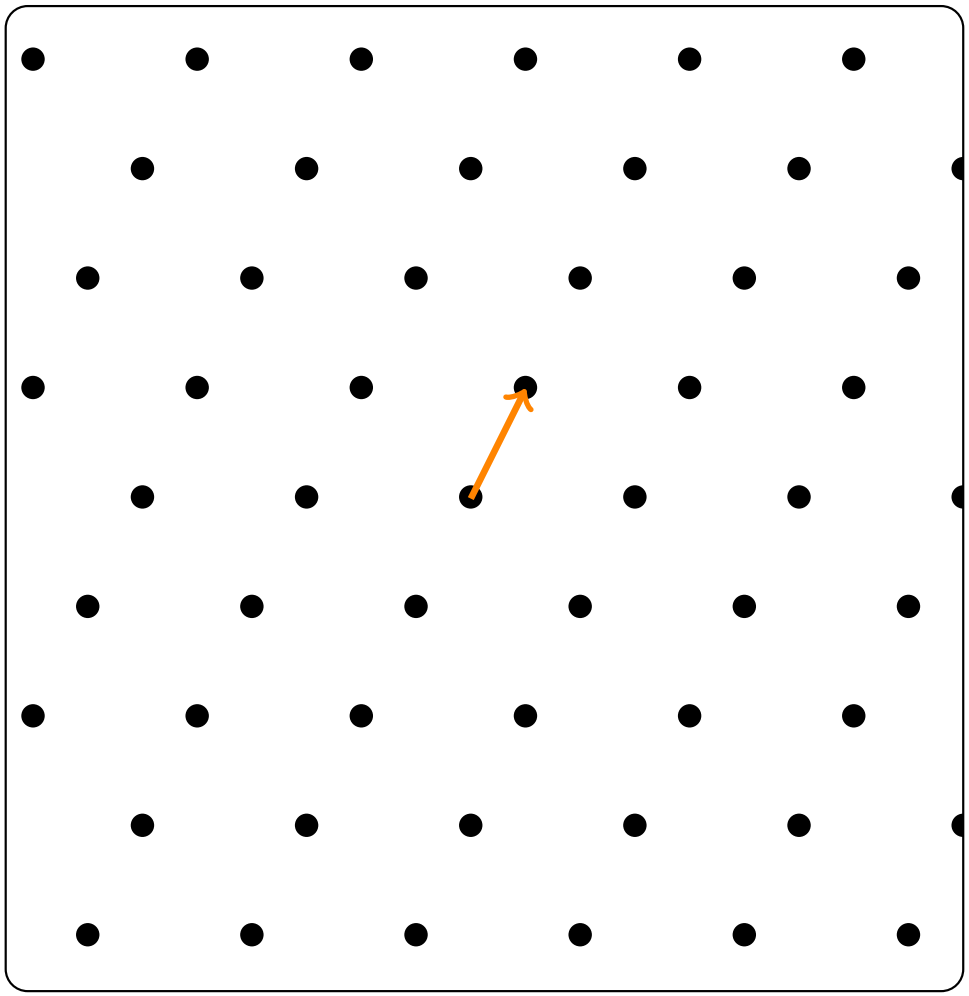

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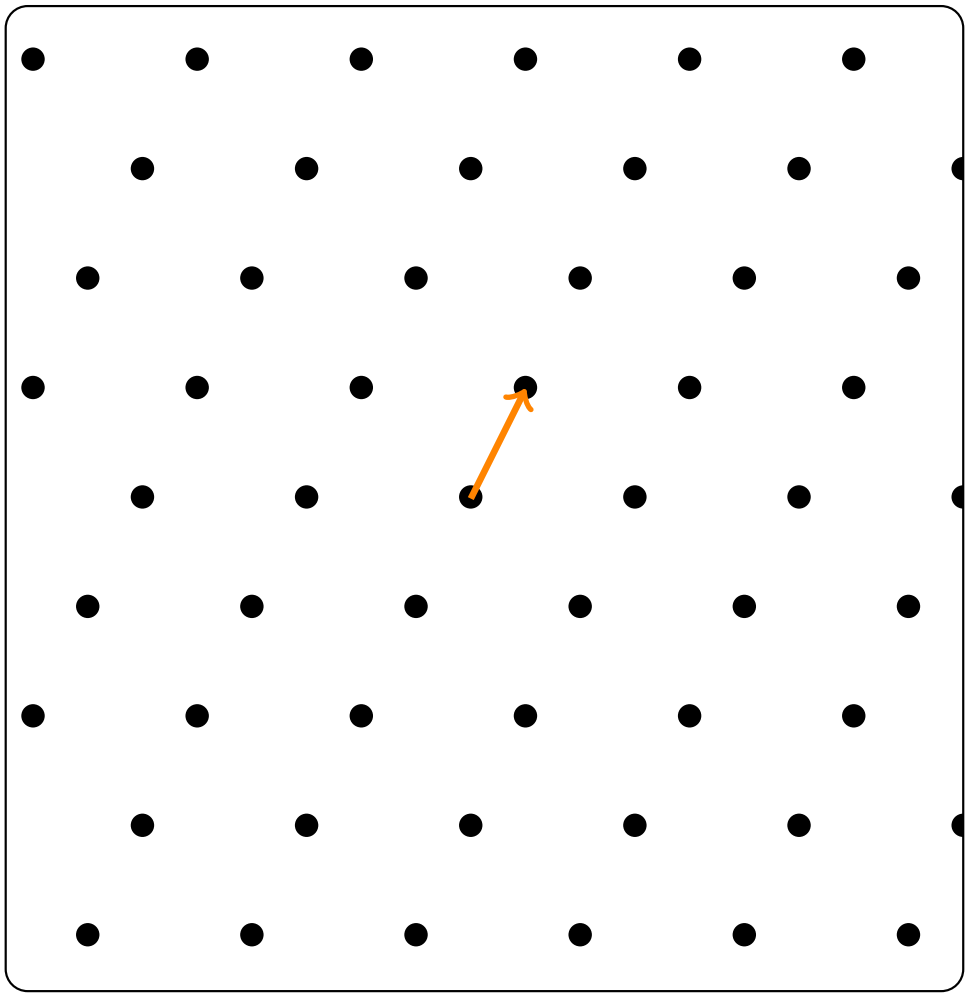

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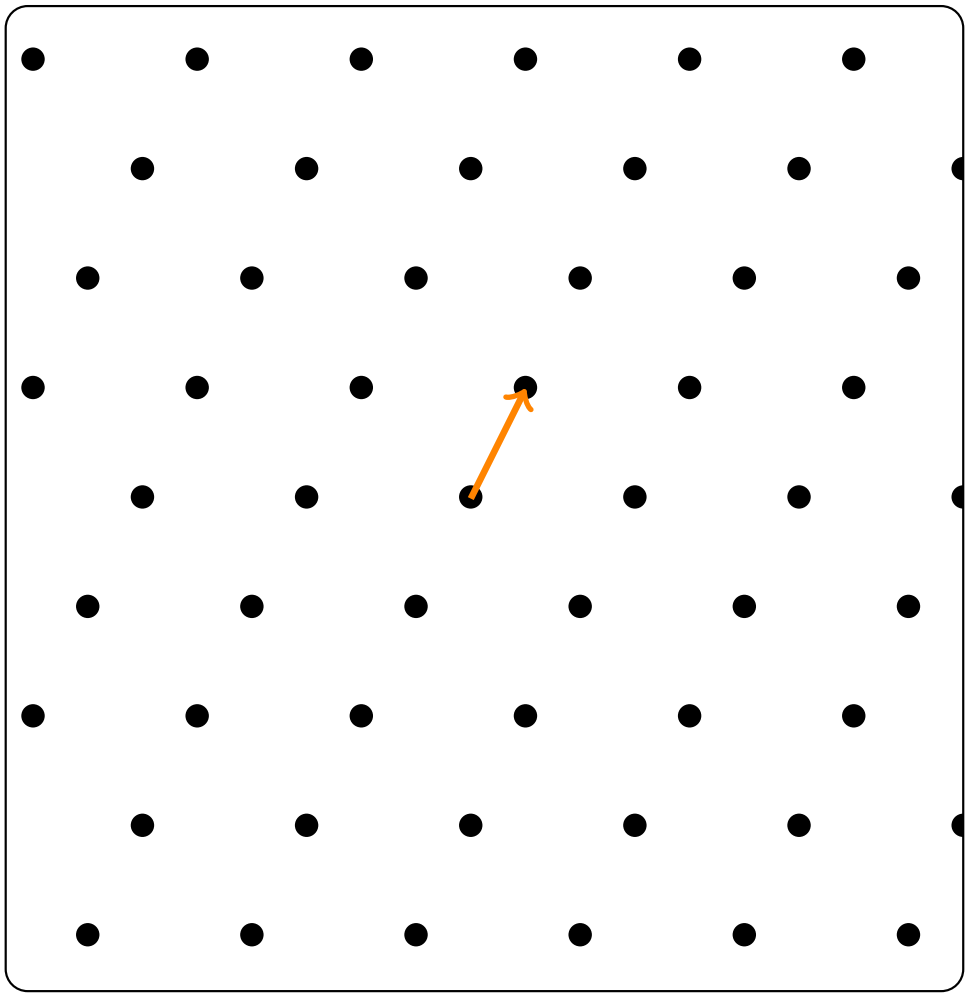

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- runs in time $2^{O(n)}$

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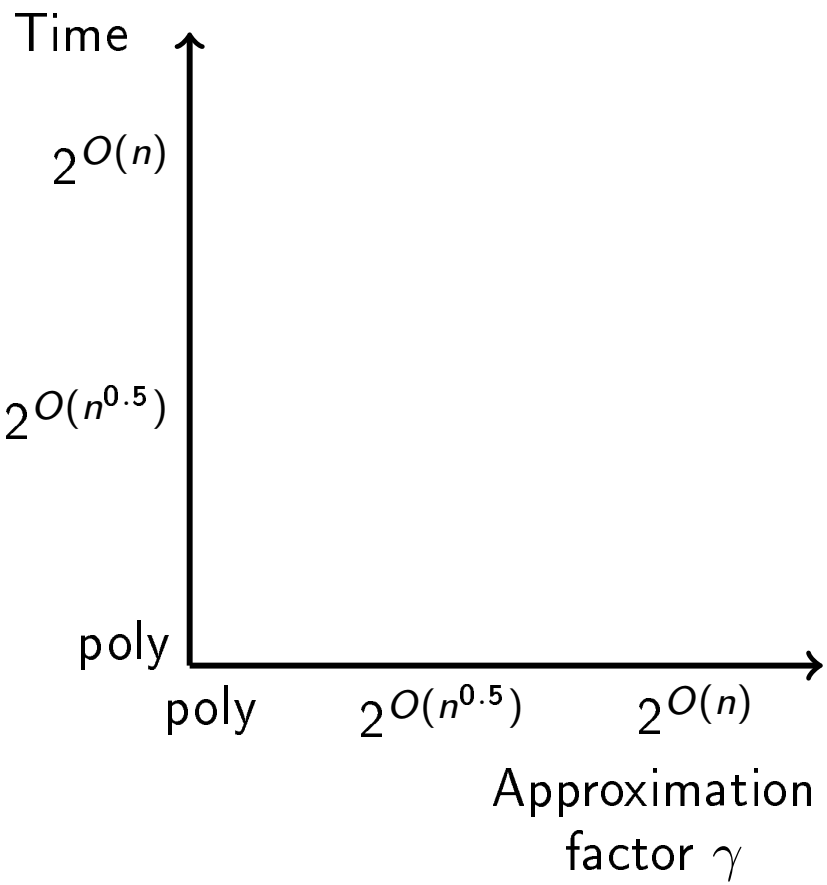
Summary and BKZ



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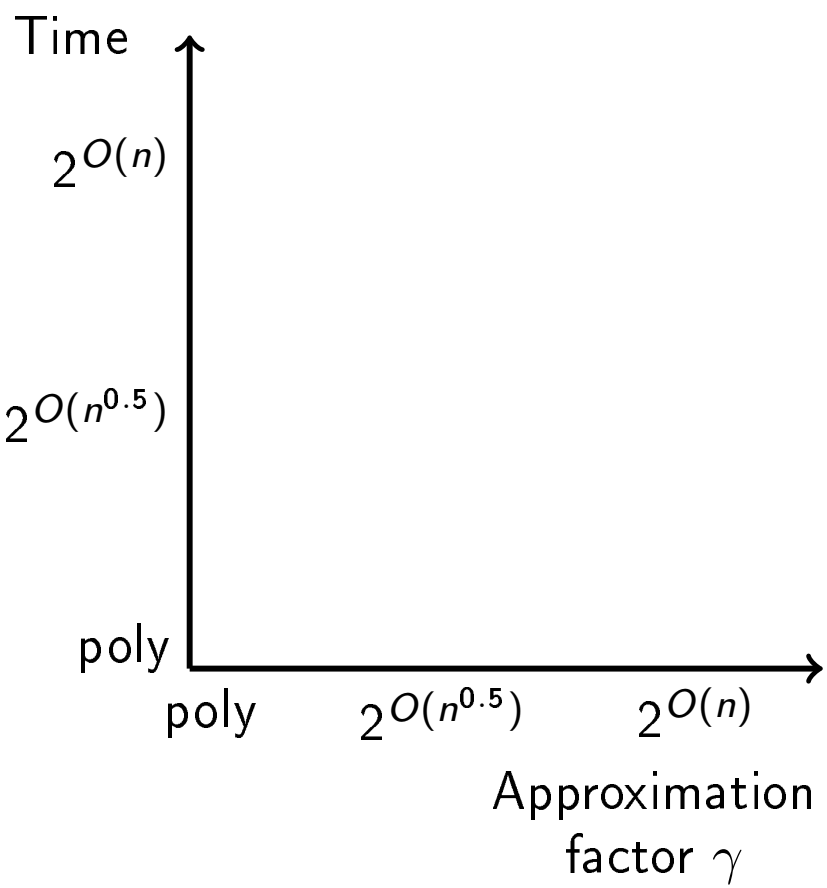

Lagrange-Gauss algorithm: dim 2

- shortest basis
- polynomial time

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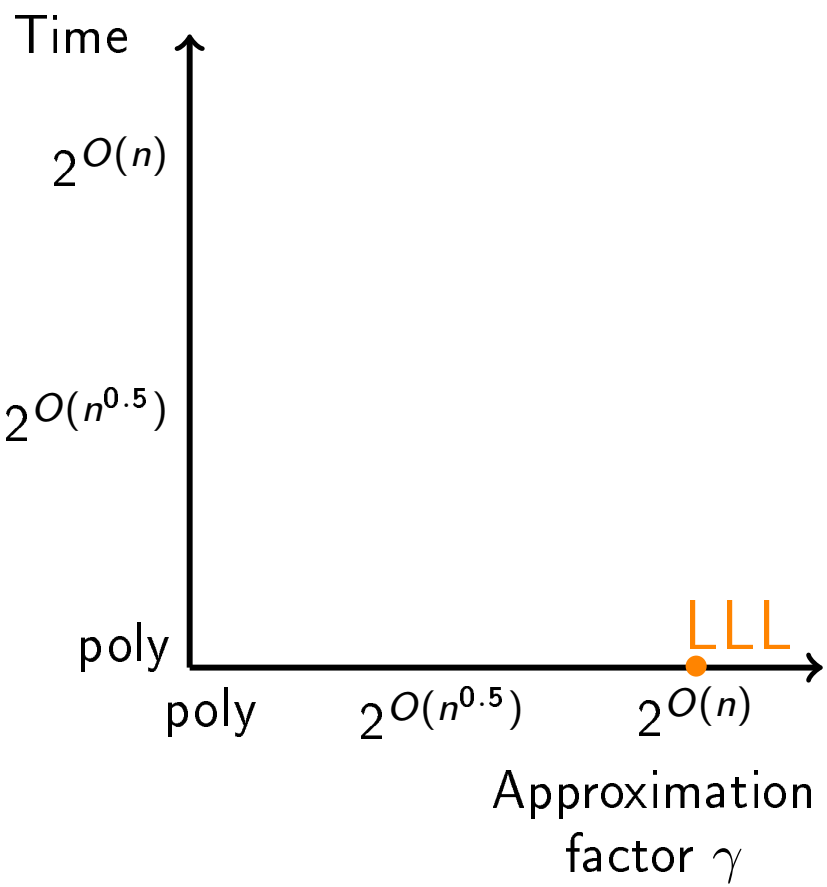

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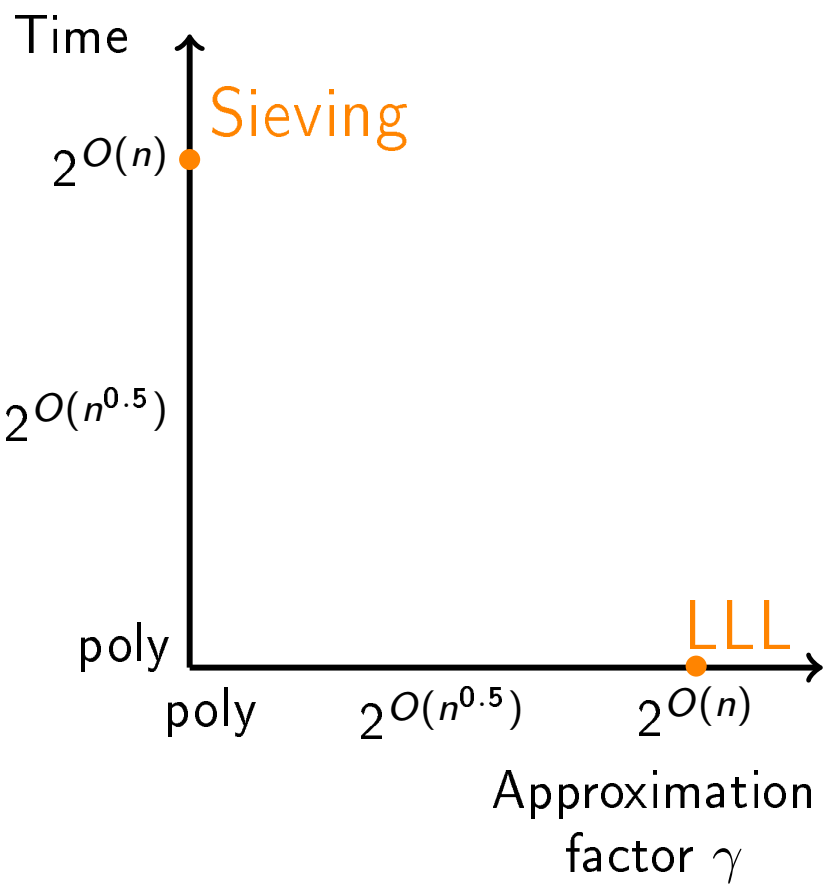

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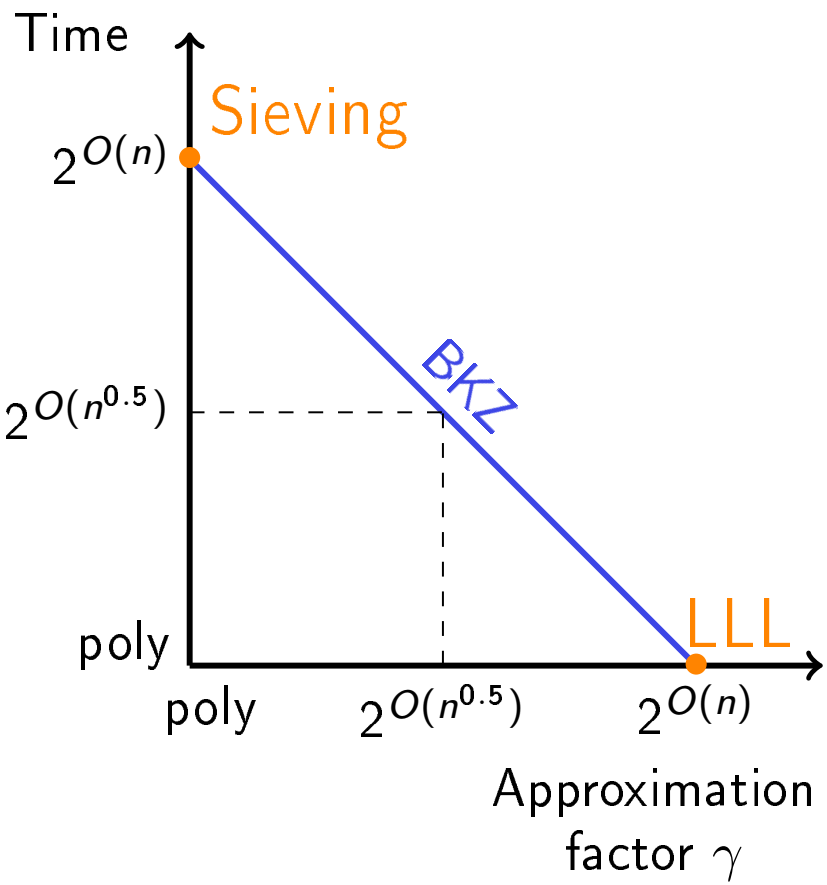
BKZ trade-offs



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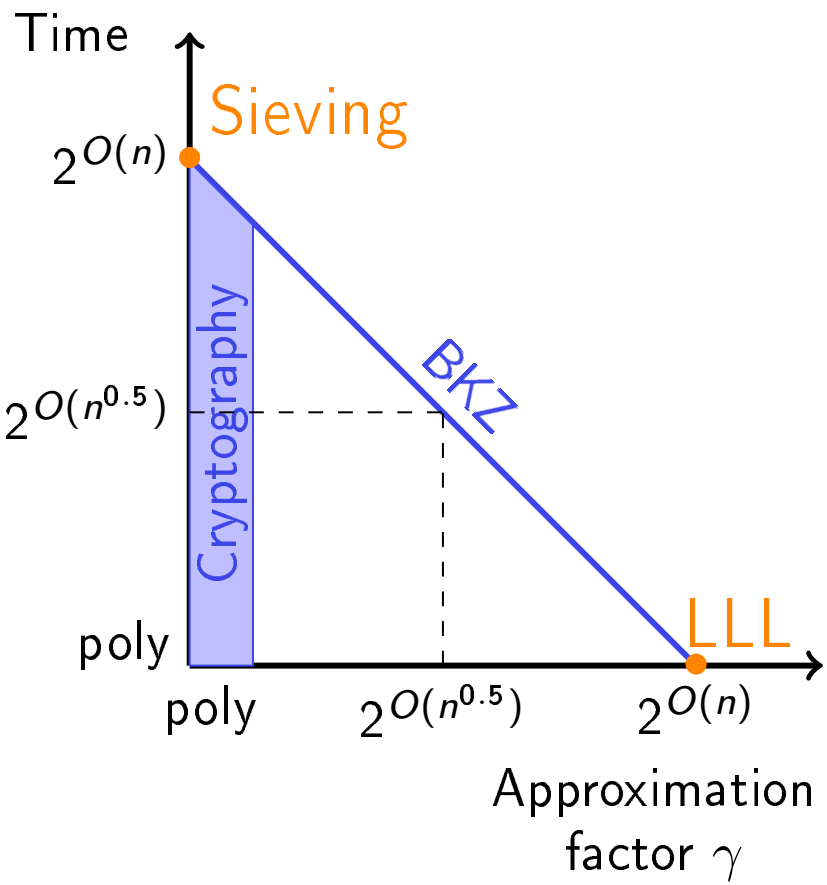
BKZ algorithm: combine LLL + Sieving $\Rightarrow$ various trade-offs

Summary and BKZ

BKZ trade-offs



Lagrange-Gauss algorithm: dim 2

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Section's conclusion

Takeaway: If the dimension n is large enough (e.g., $n \geq 1000$) computing short bases of lattices with approximation factor $\gamma \lesssim n$ is very hard.

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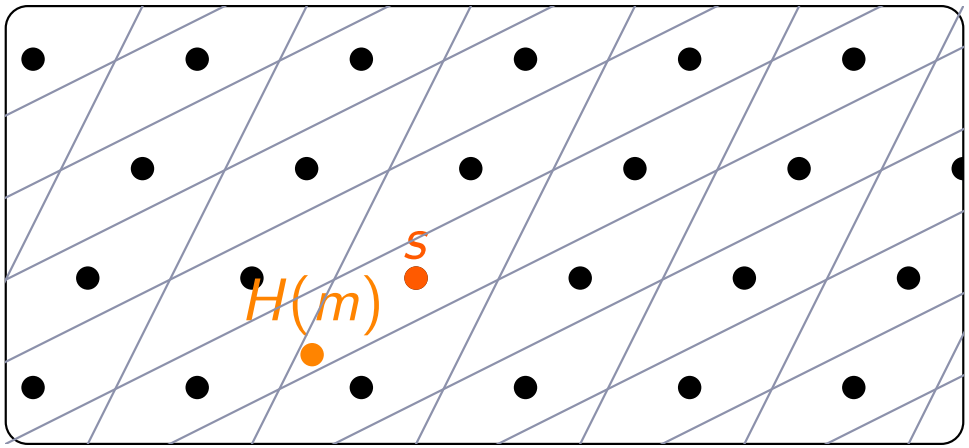
- It is hard if you want an algorithm that works on all lattices
- It may be easy for some lattices!
  (We will come back to this...)

Constructing signatures from lattices

(using hash-and-sign technique)



Digital Signature

Three algorithms:

- $\mathsf{KeyGen}() \rightsquigarrow (pk, sk)$
- $\mathsf{Sign}(m, sk) \rightsquigarrow s$
- $\mathsf{Verify}(m, s, pk) \rightsquigarrow$ yes/no

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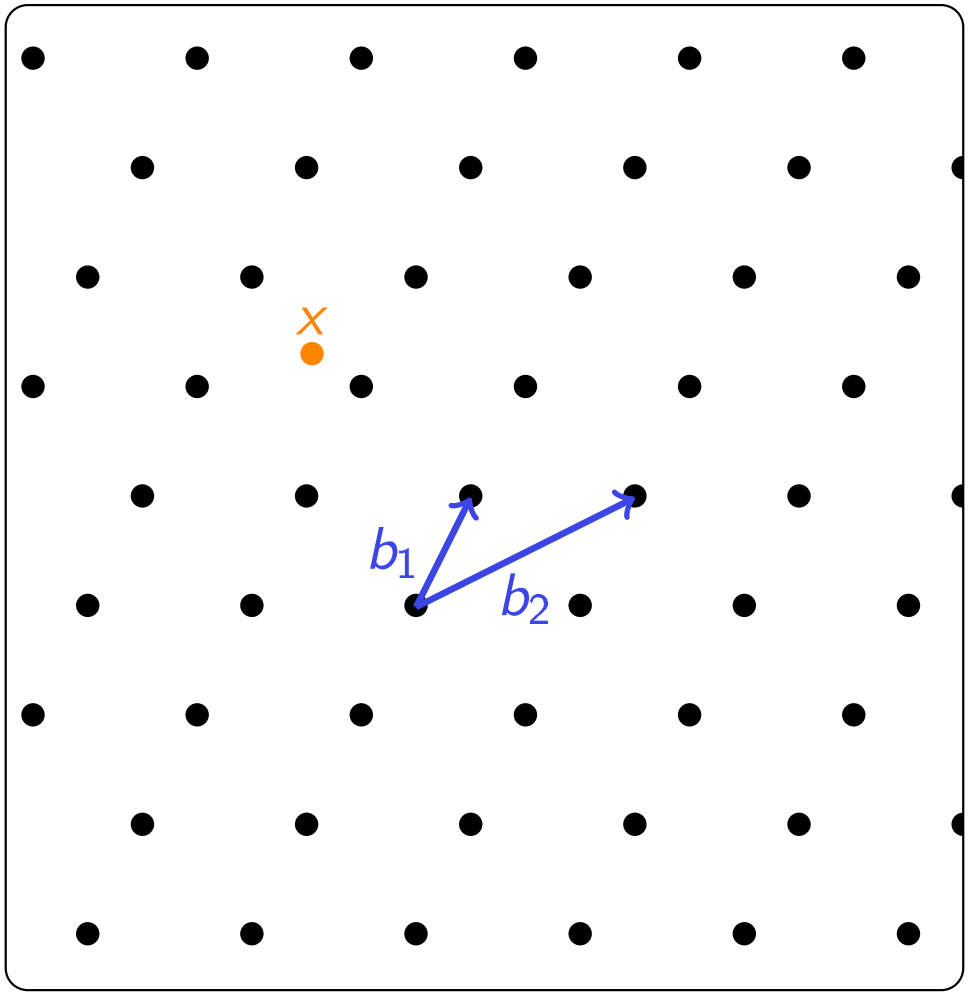
Security: an attacker not knowing sk cannot forge valid pairs (m, s)
(i.e., such that $\mathrm{Verify}(m, s, pk) = \mathrm{yes}$)

Decoding in a lattice using a short basis



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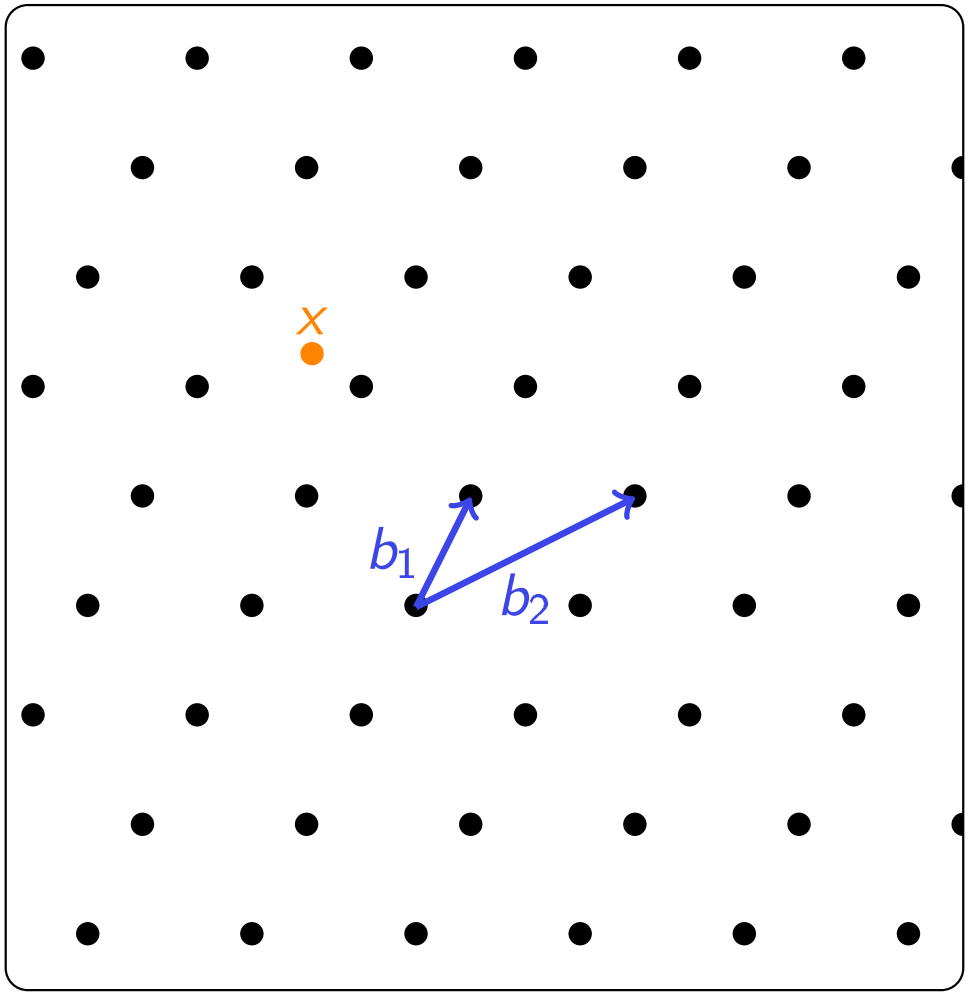

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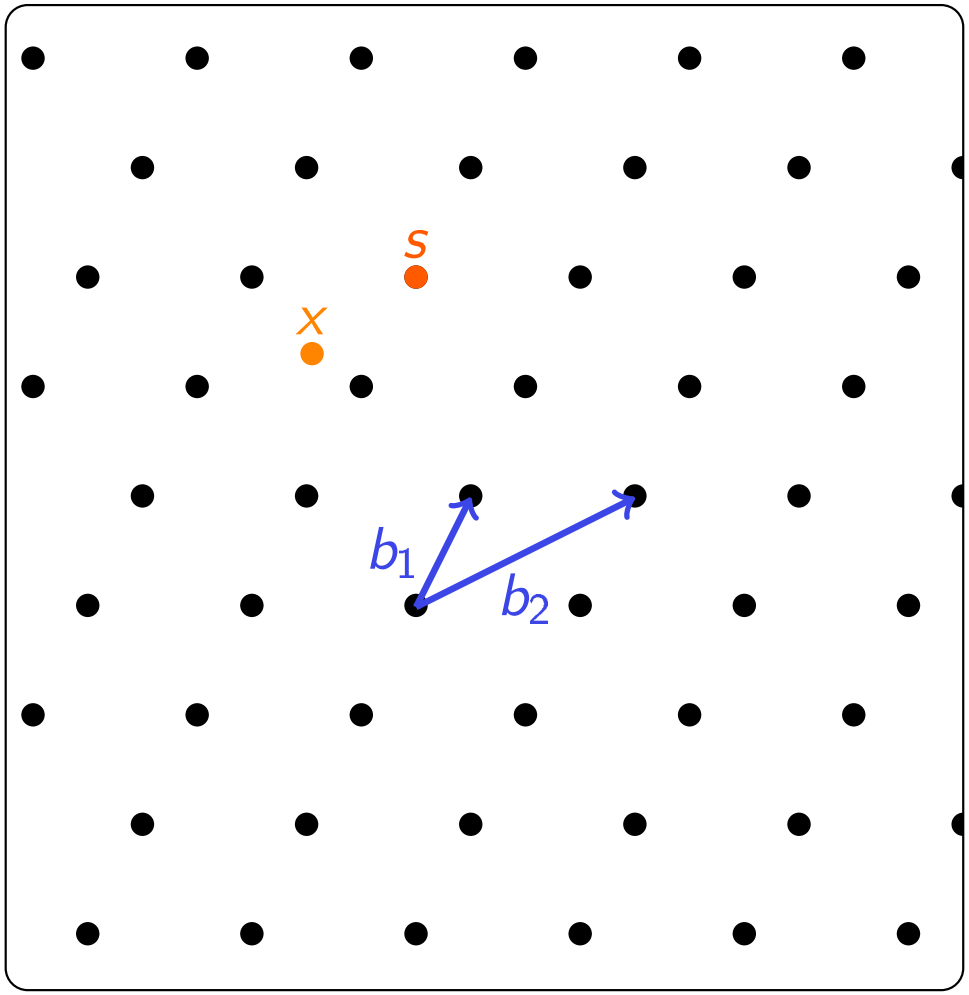

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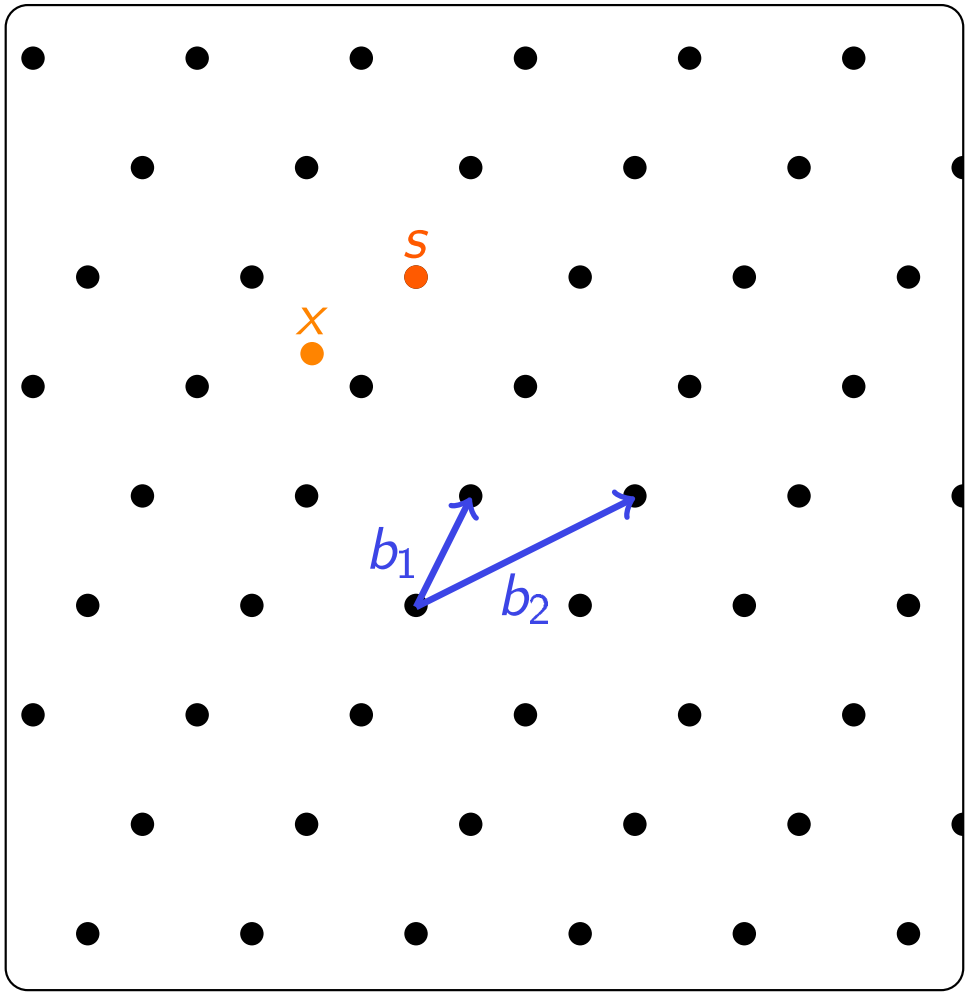

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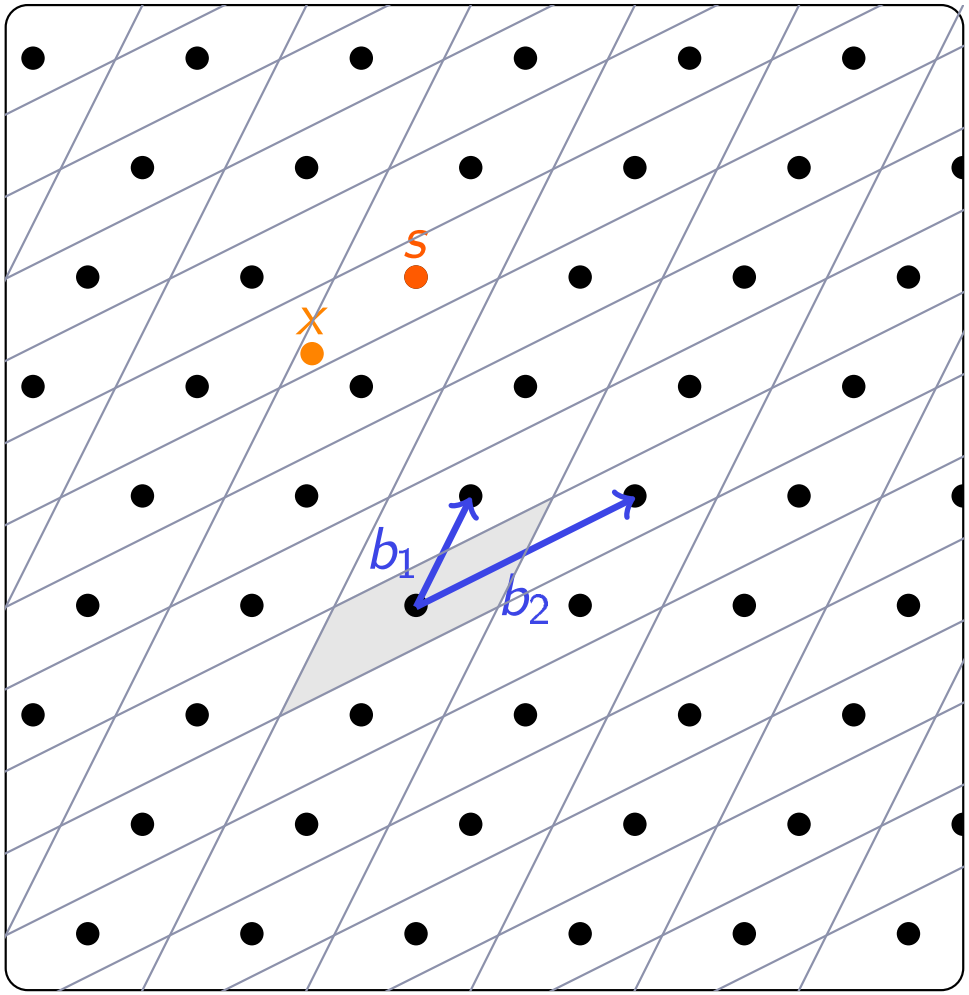

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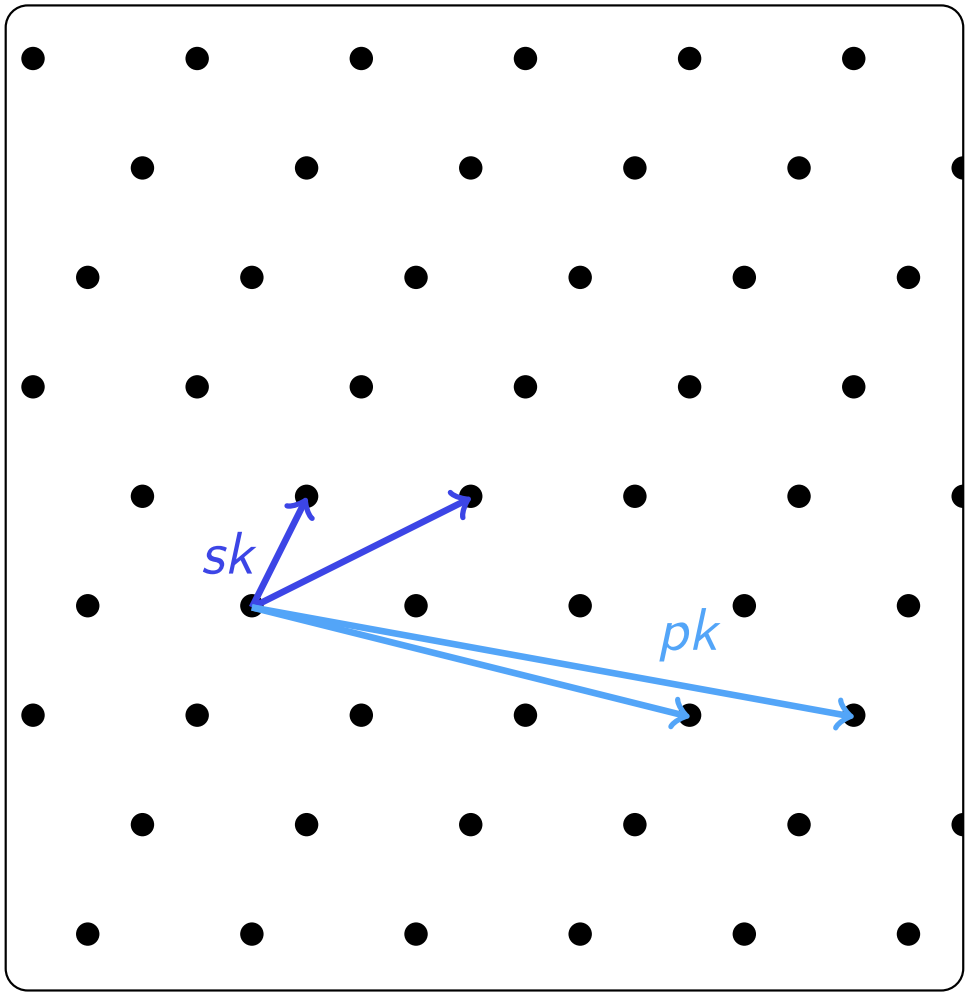
Hash-and-sign: first idea [GGH97]



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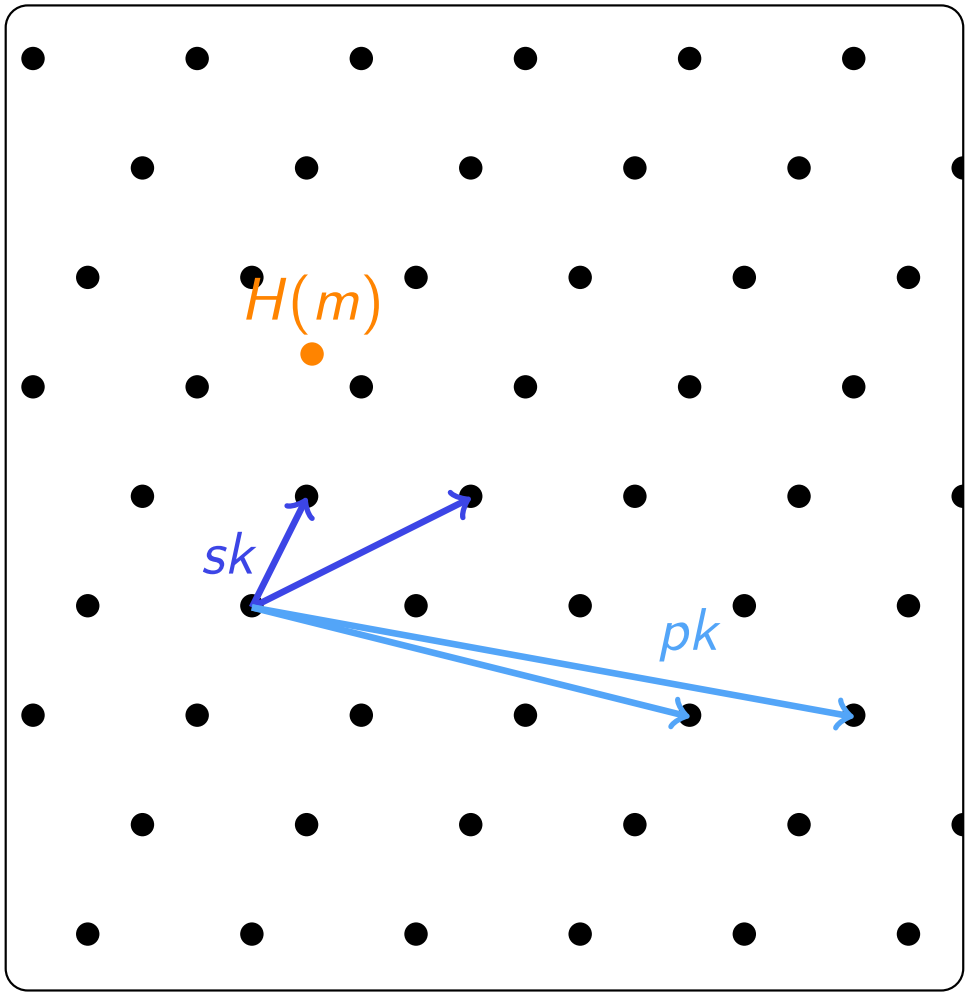

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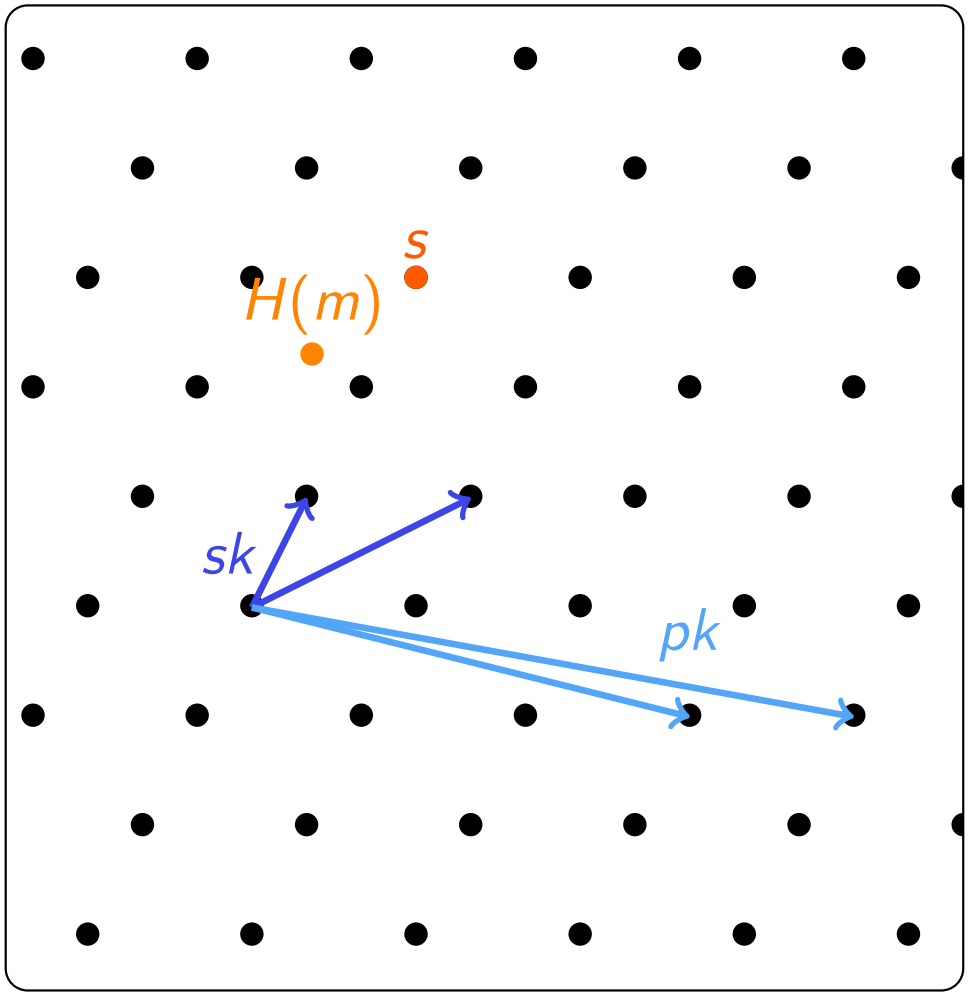

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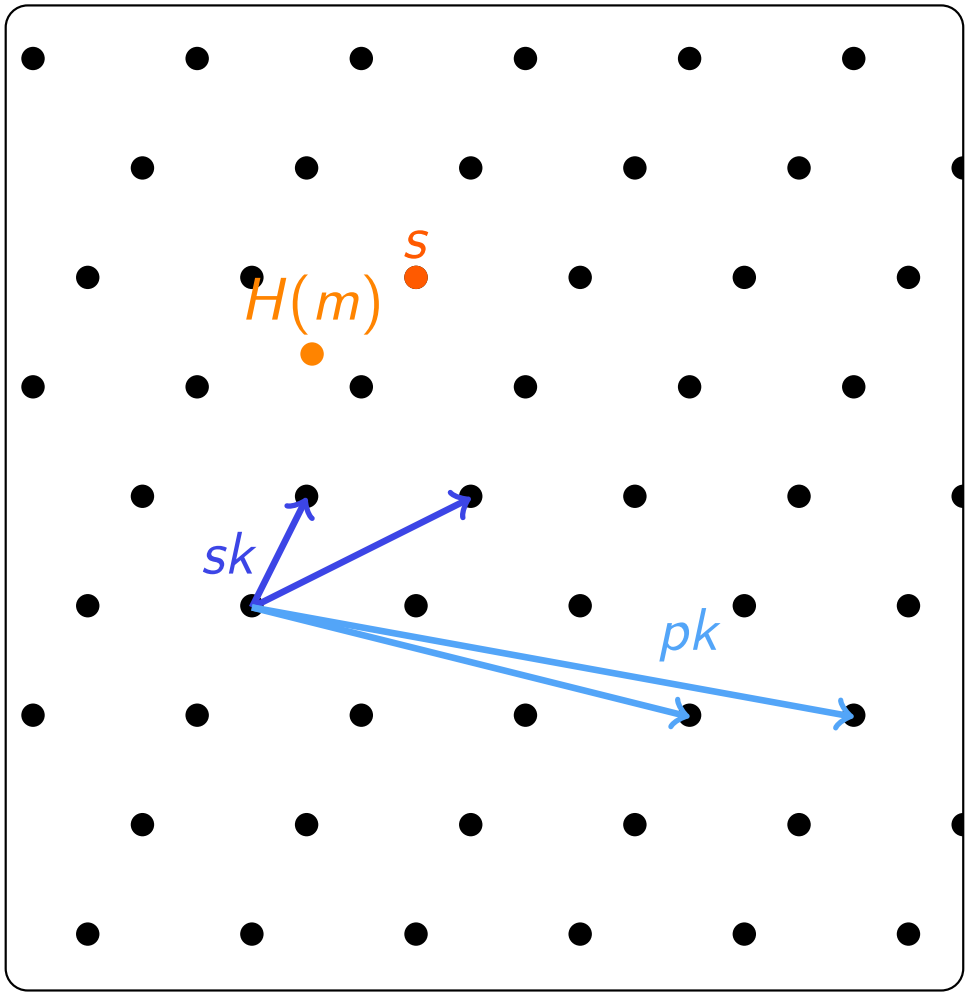

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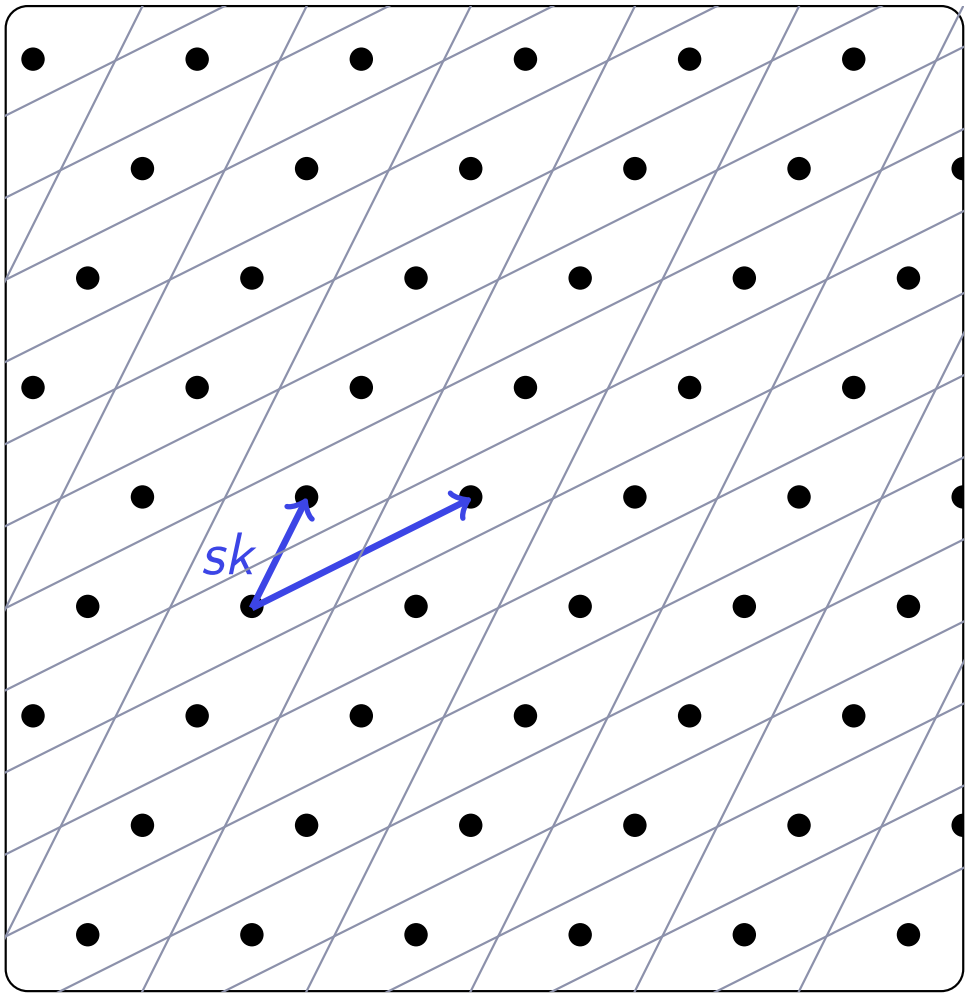

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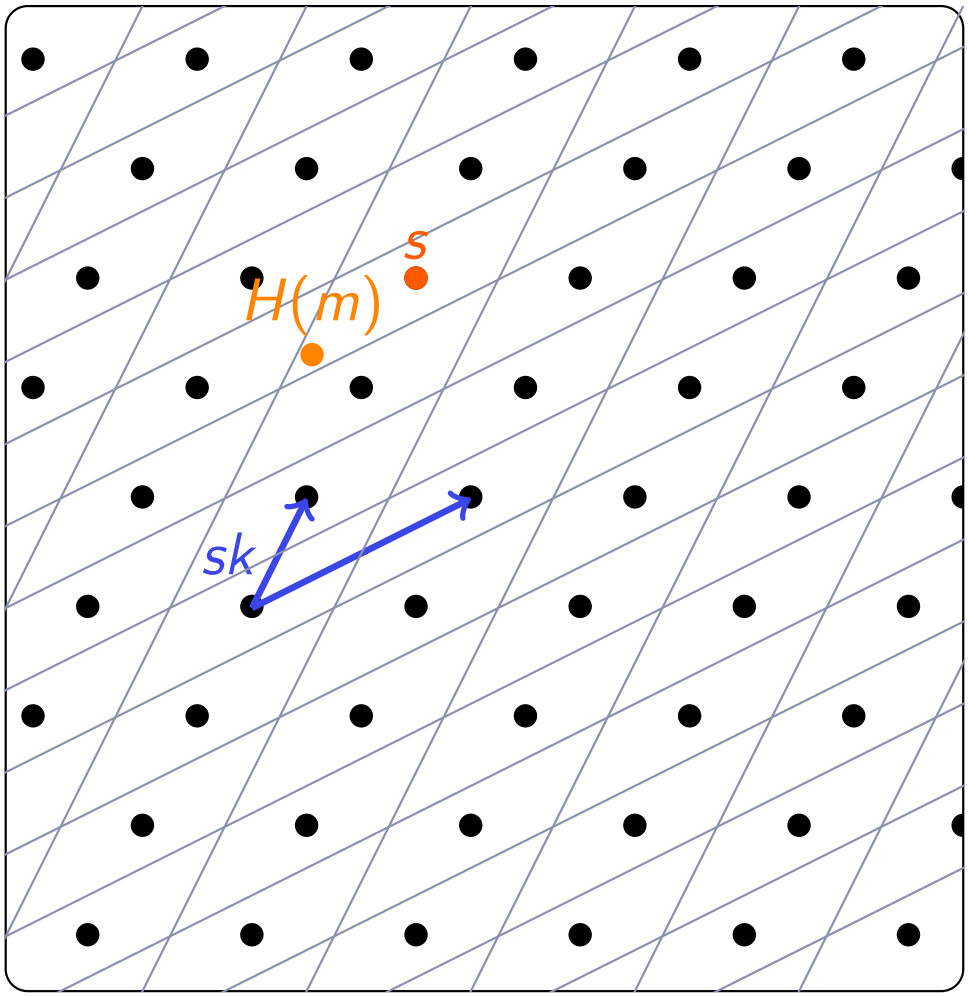

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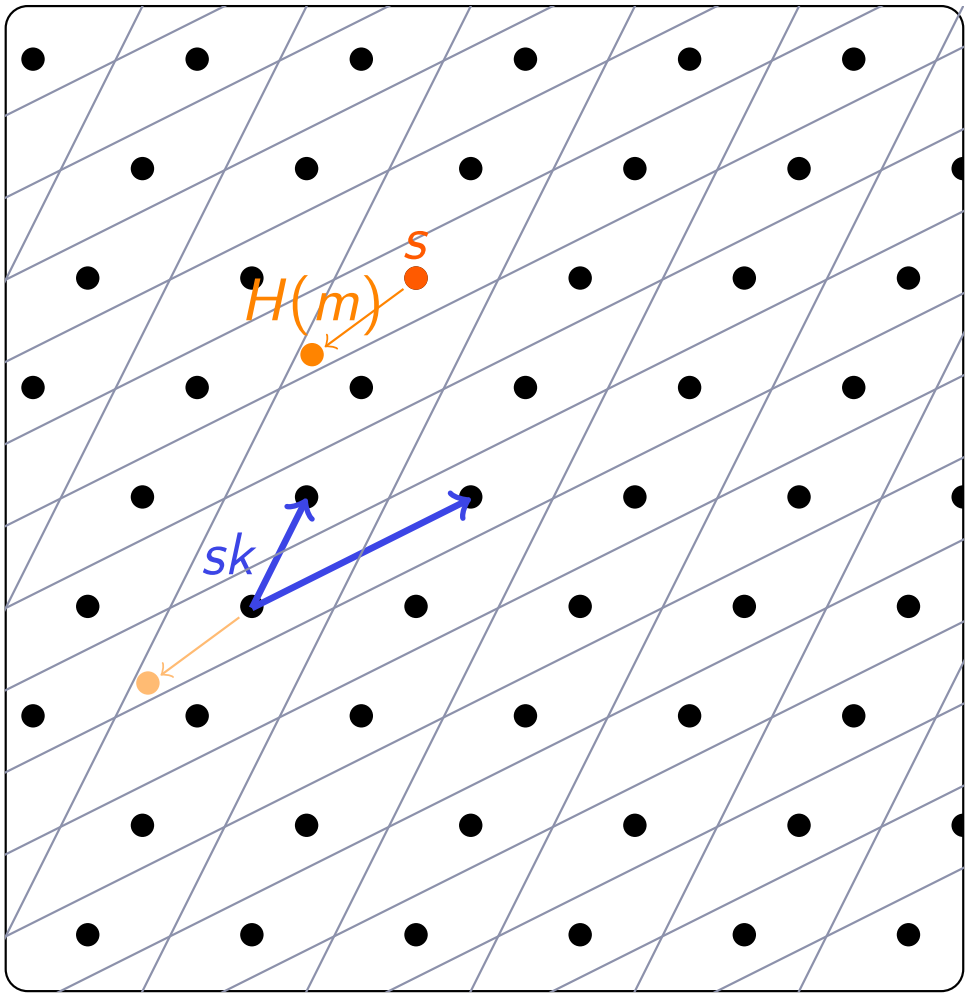

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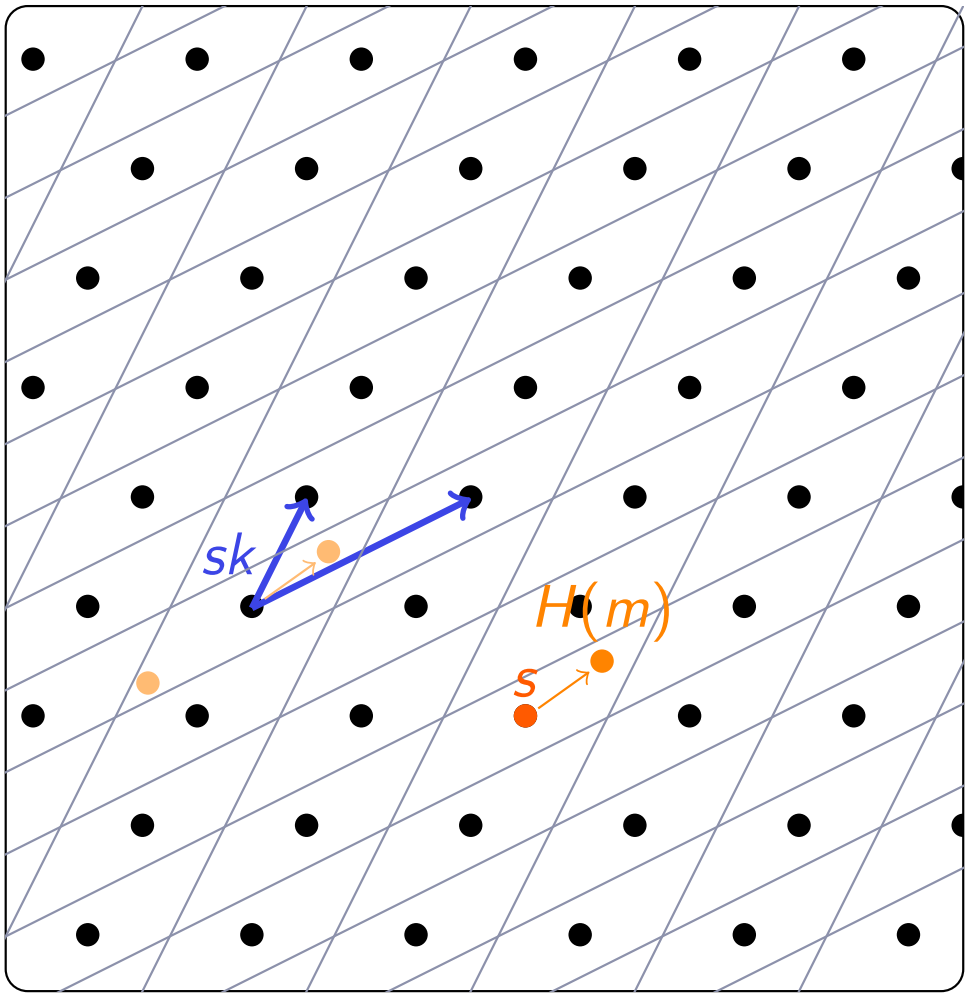

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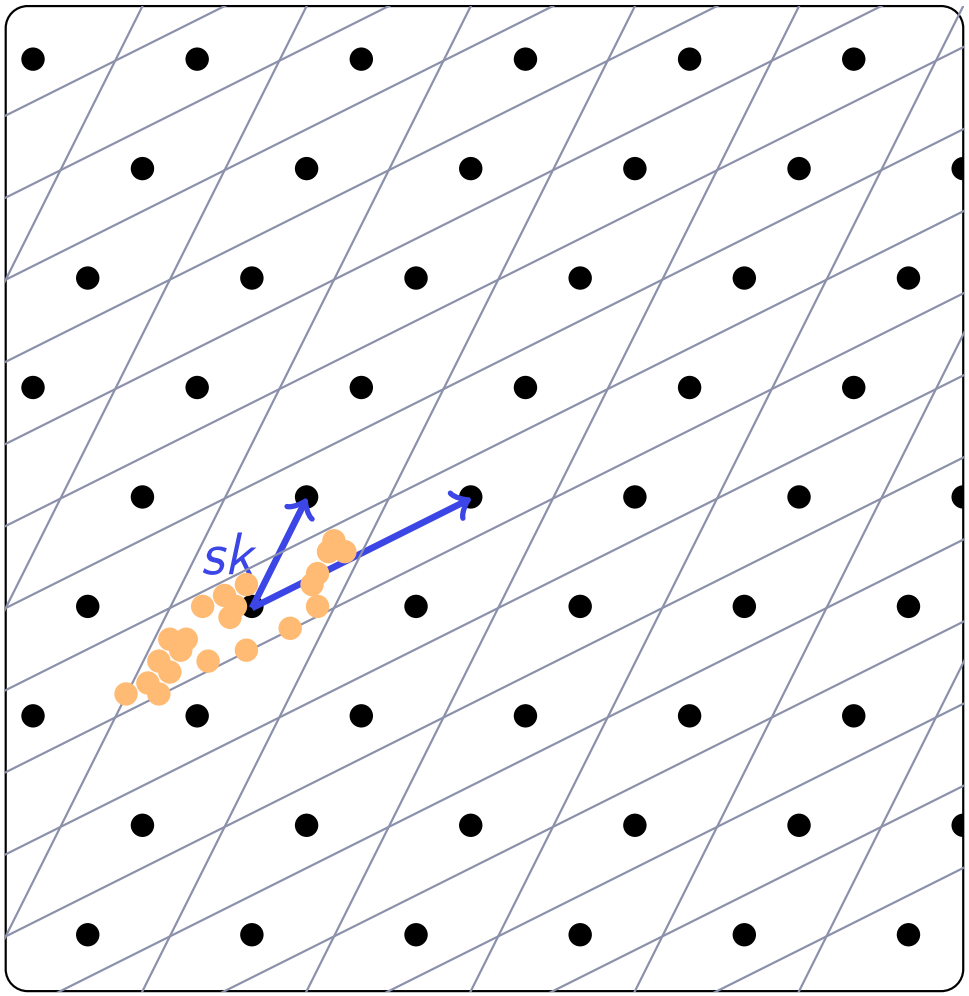

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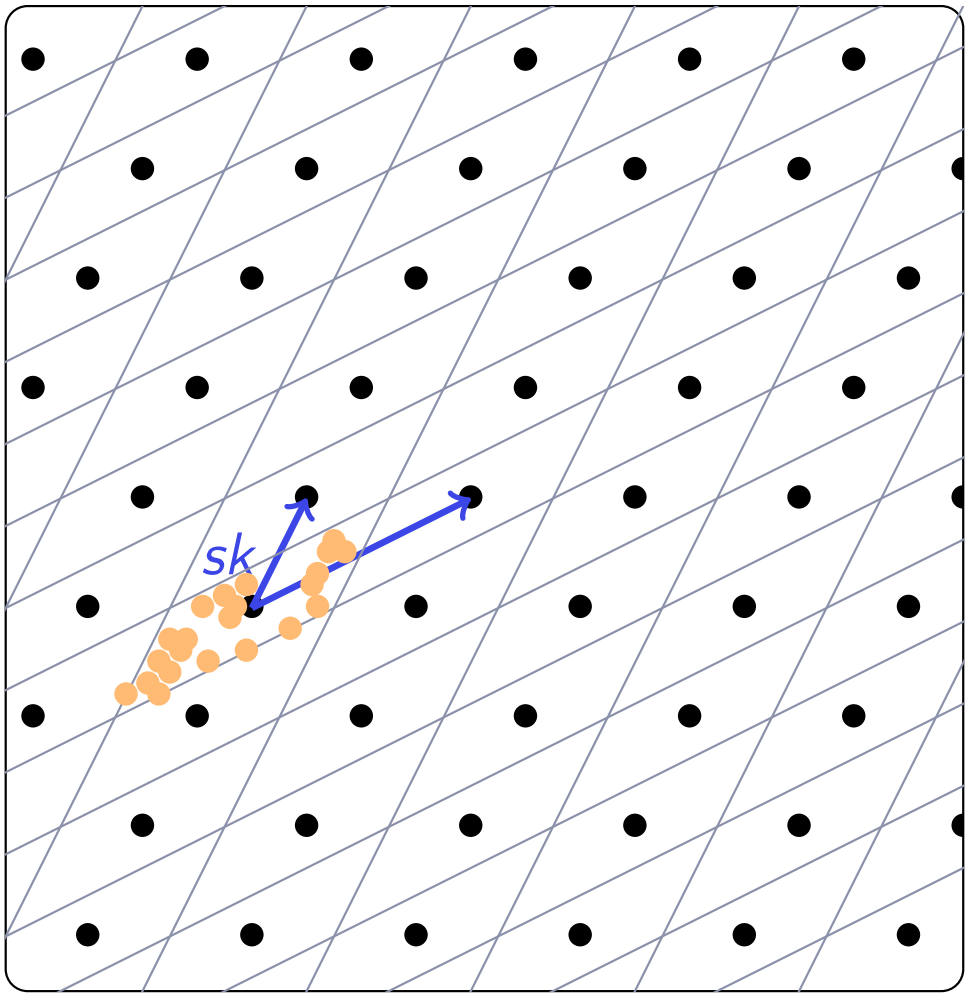

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From the shape of the parallelepiped, one can recover the short basis

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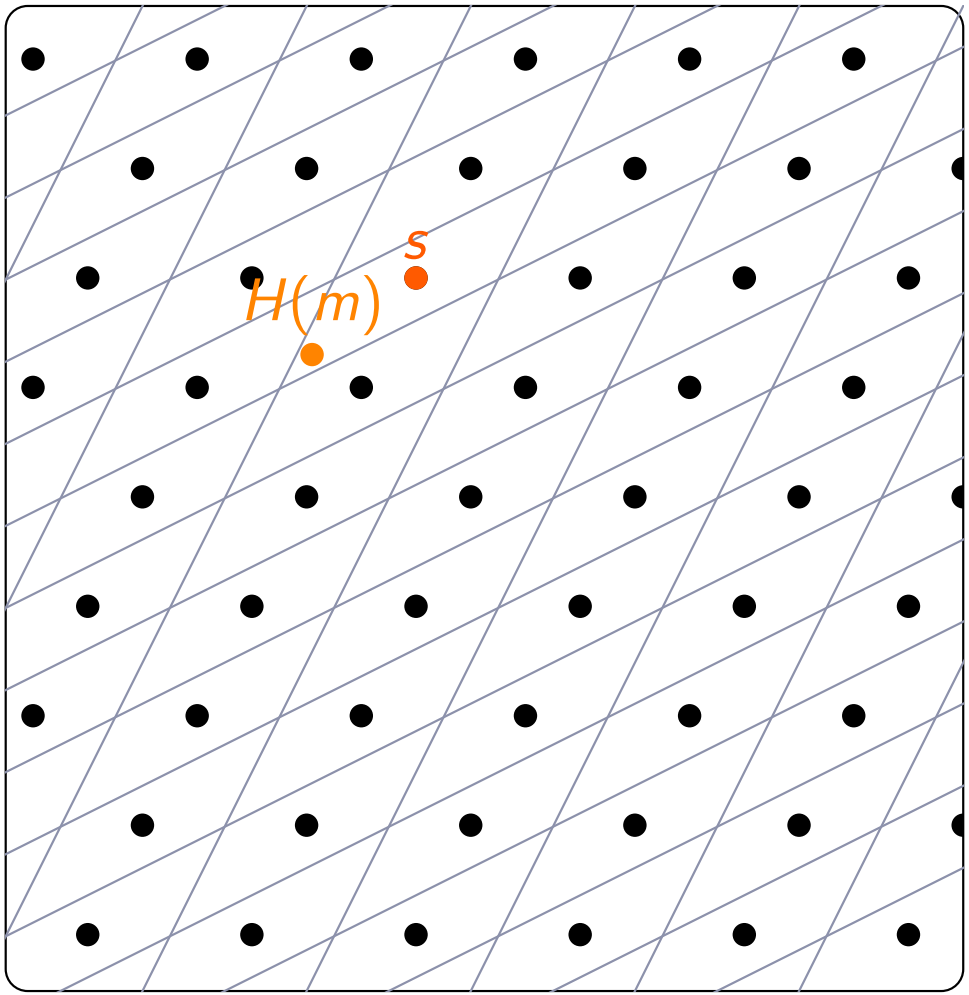
Preventing the attack [GPV08]



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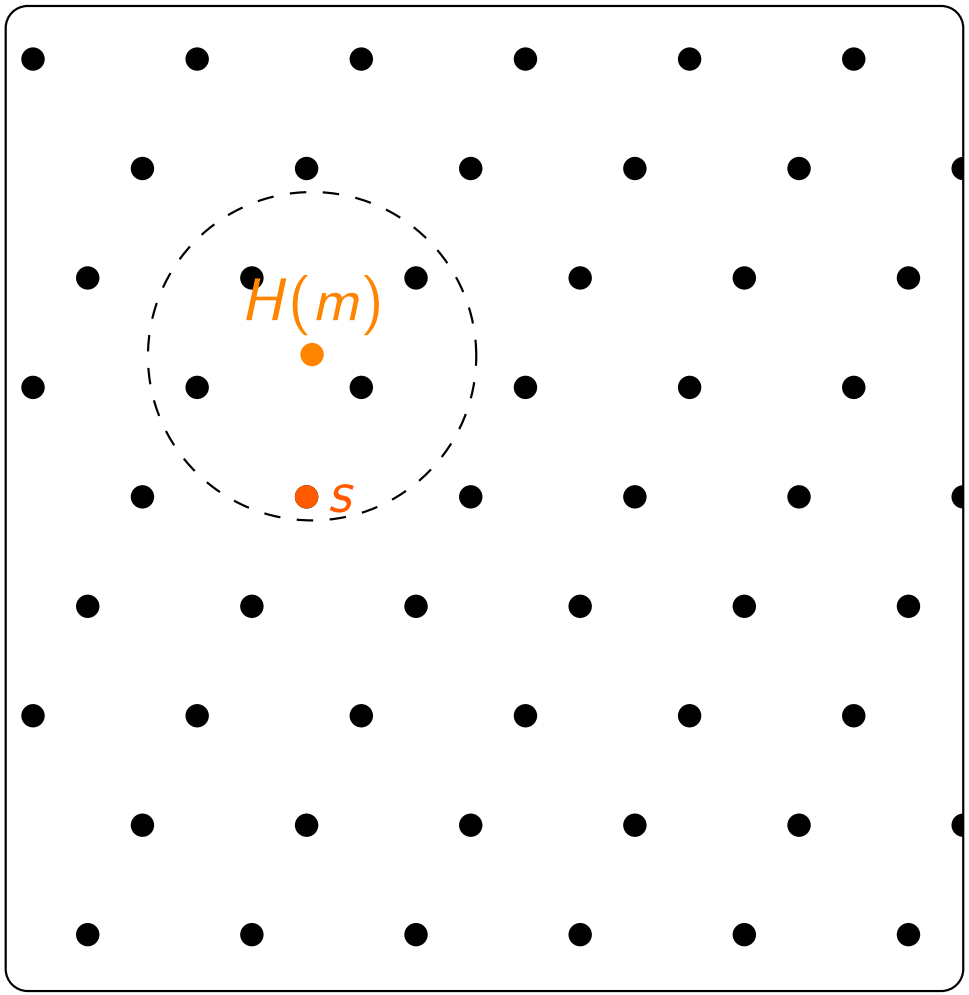

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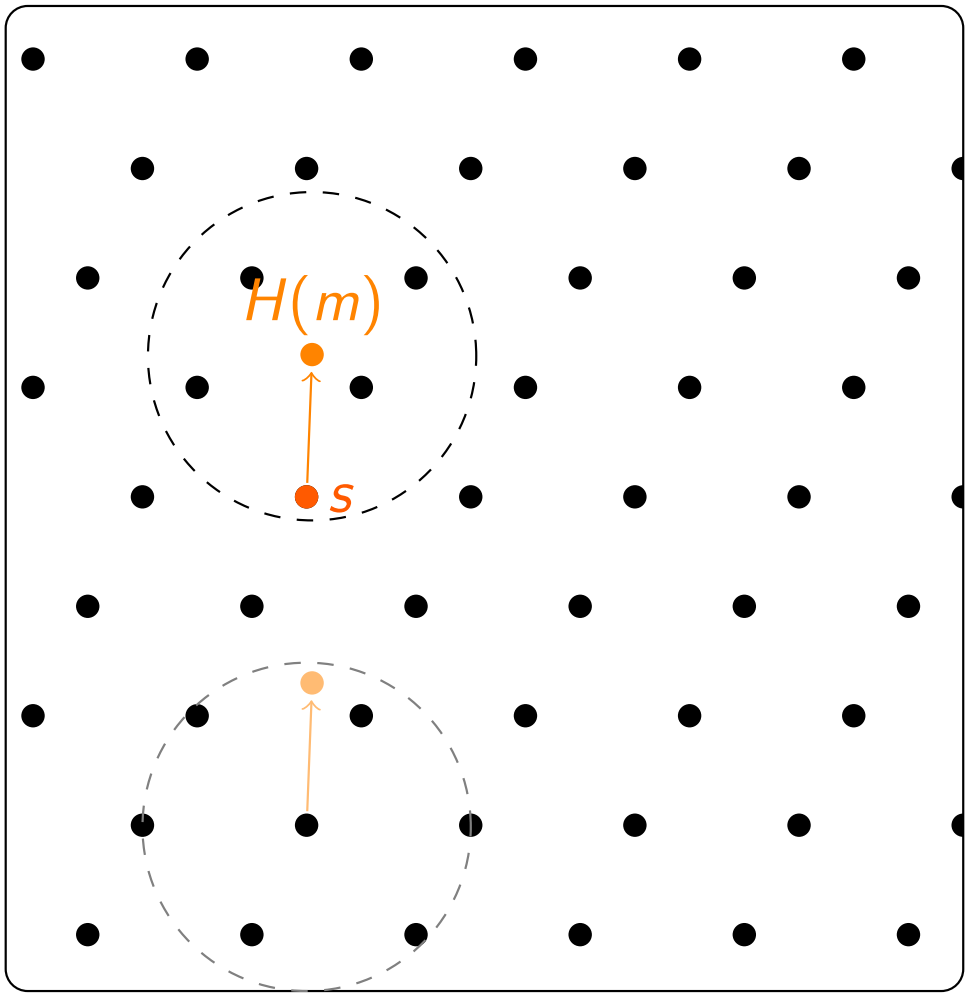

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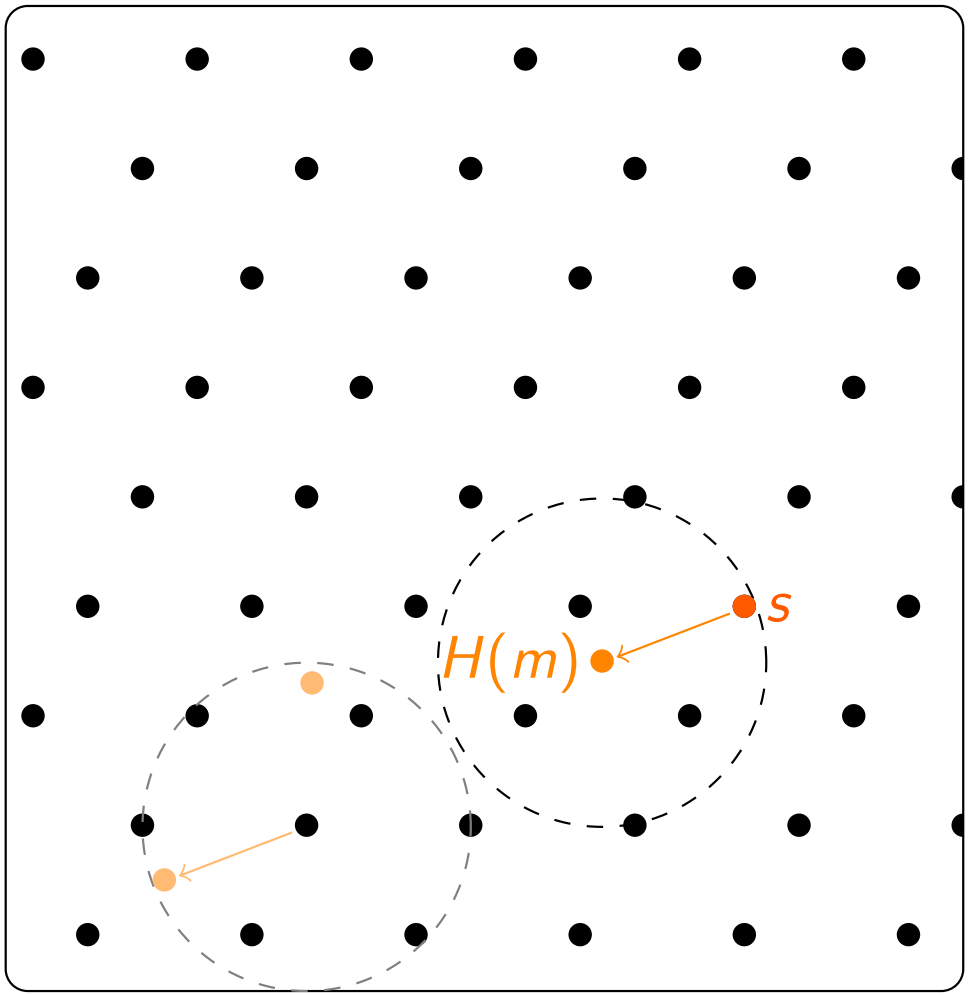

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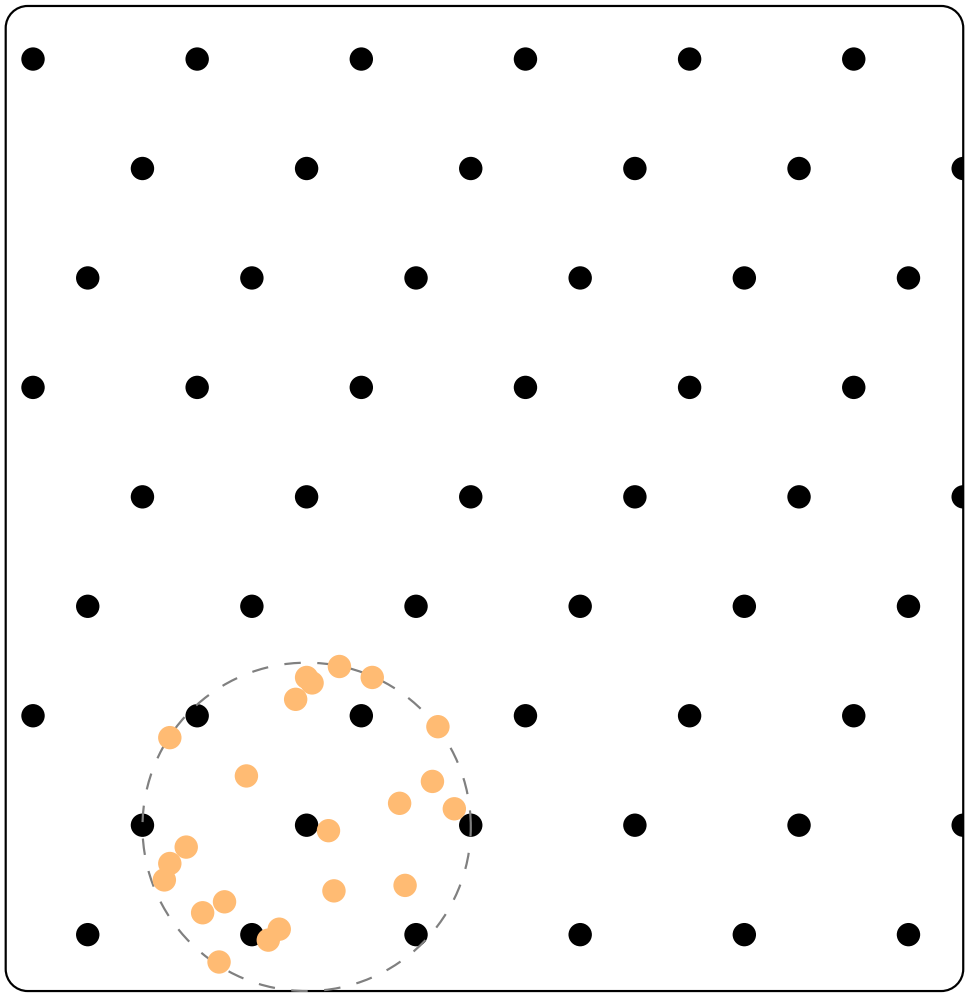

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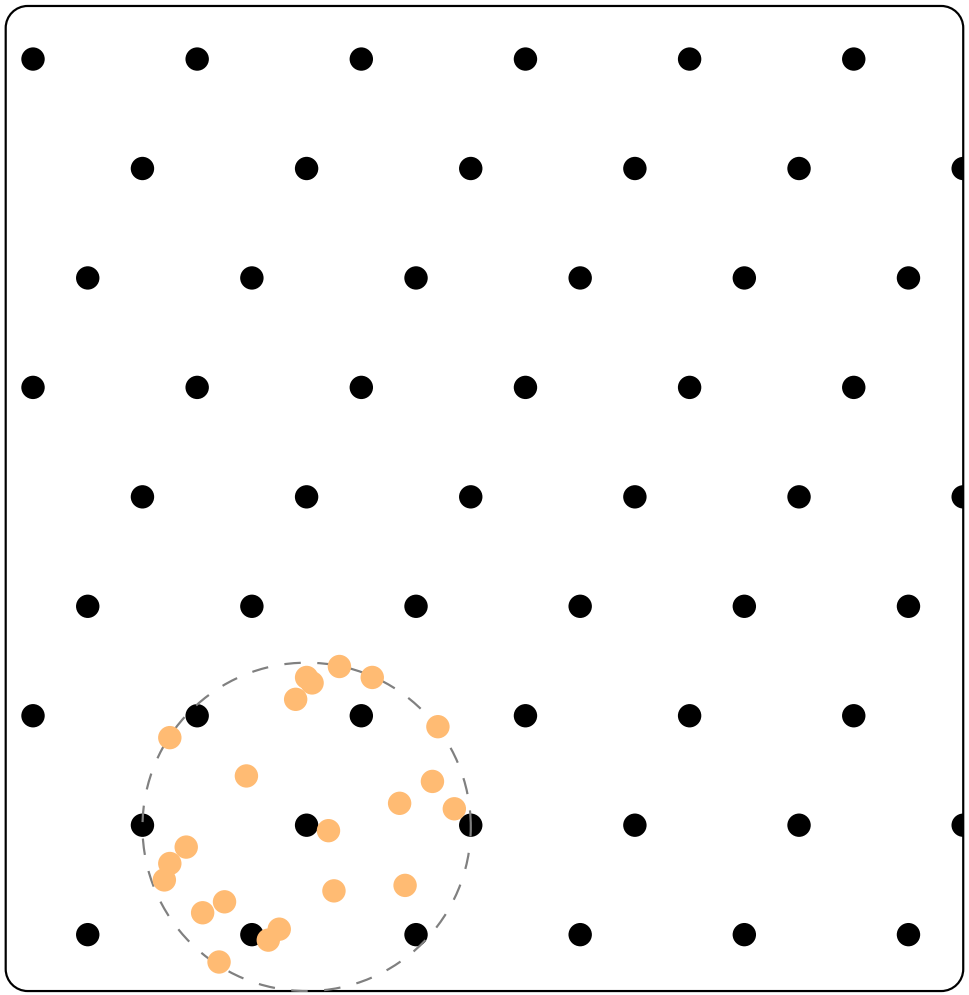

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Lemma: if an adversary can forge signatures, then she can recover a short basis of $\mathcal{L}$ using only pk (in the ROM)

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Digression: Sampling uniformly in a ball [PP21]

Input: center x, radius r

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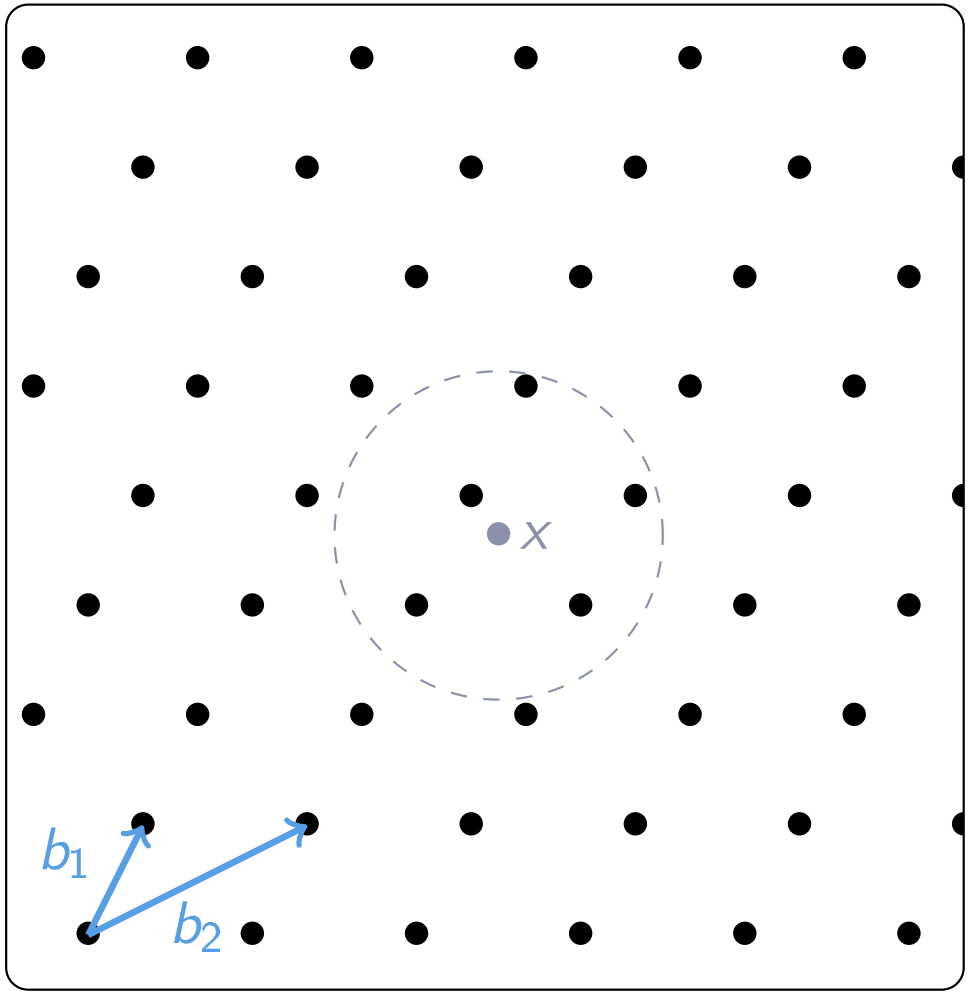

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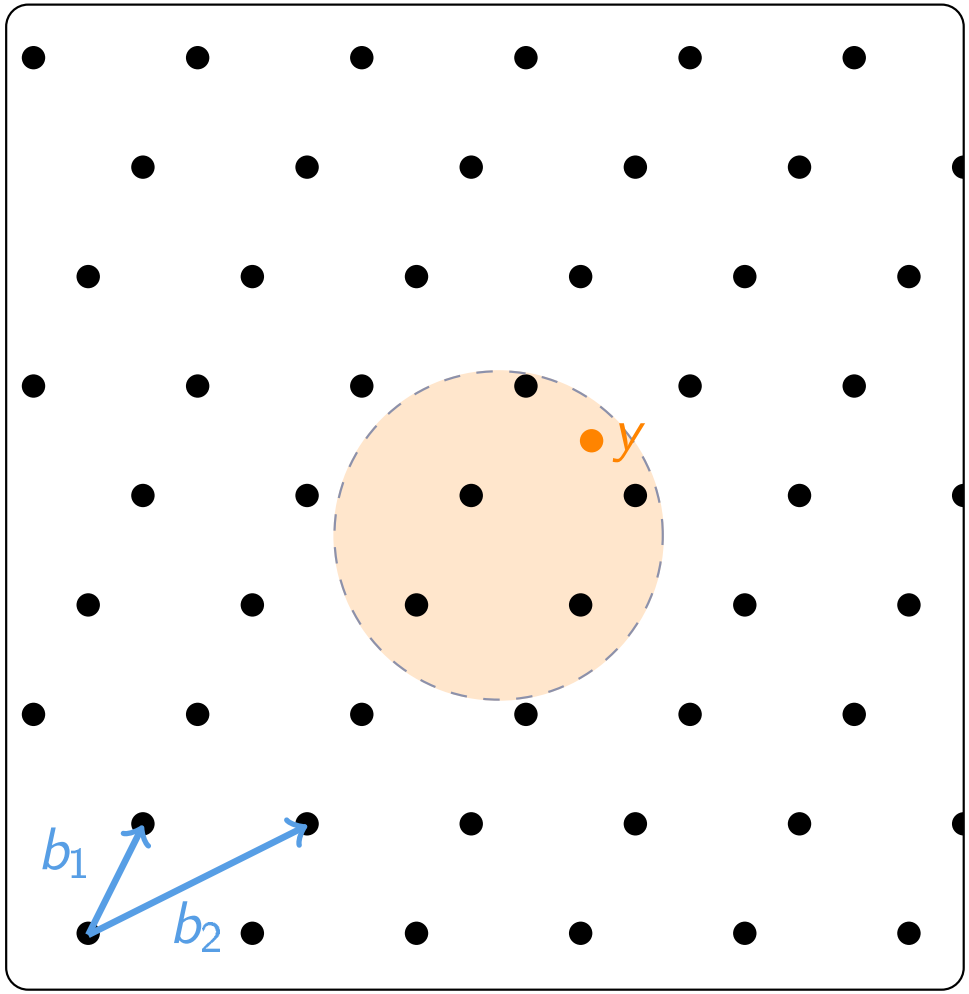

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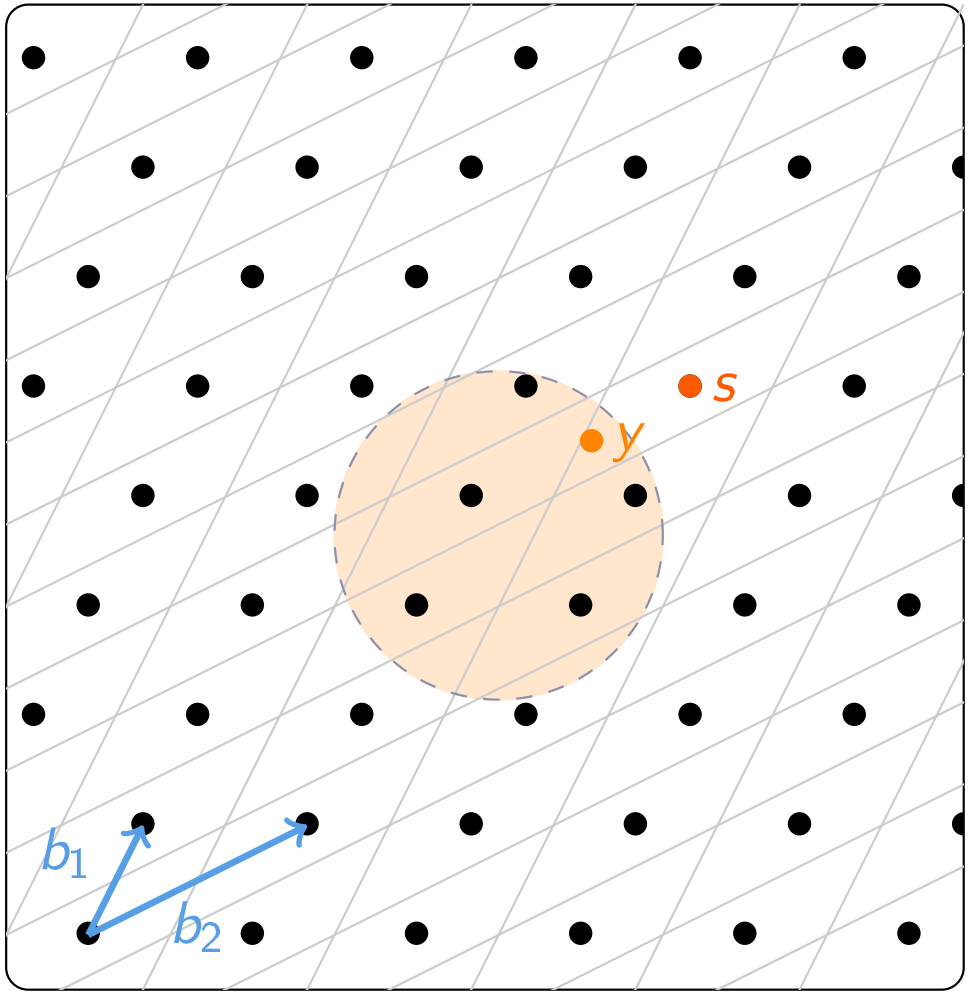

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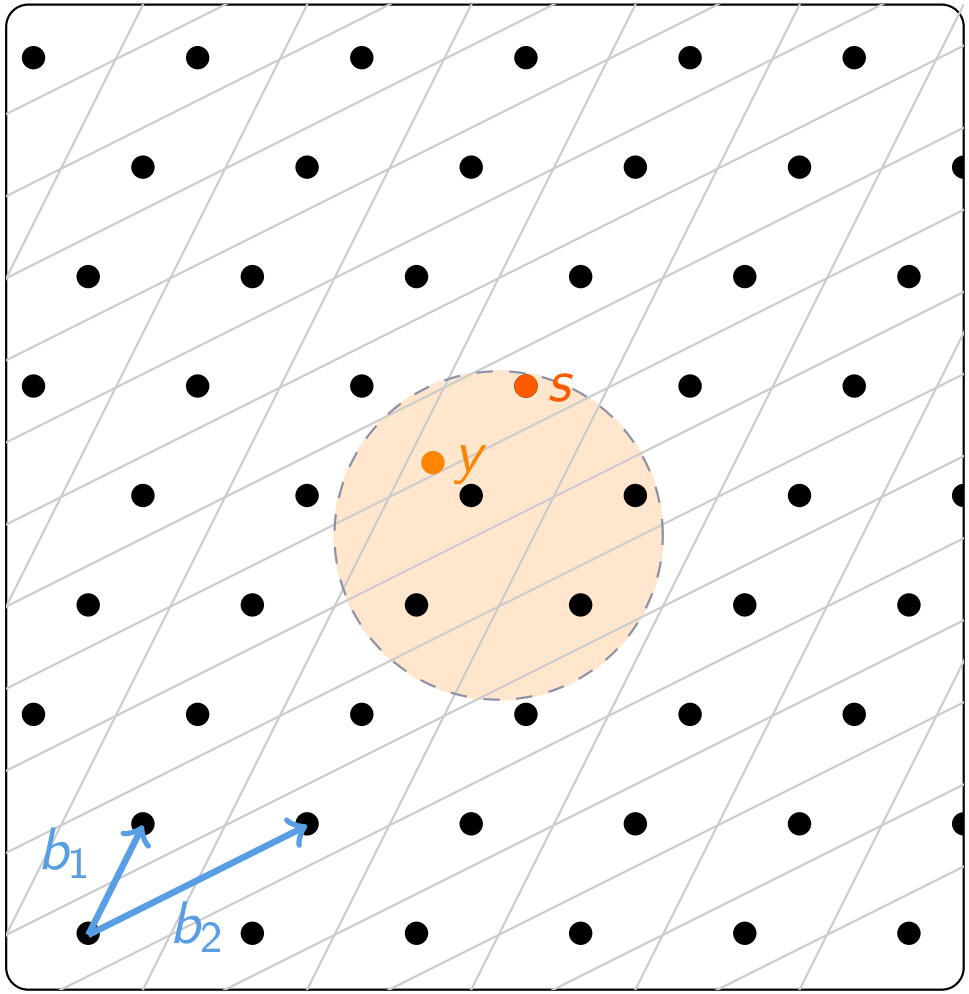

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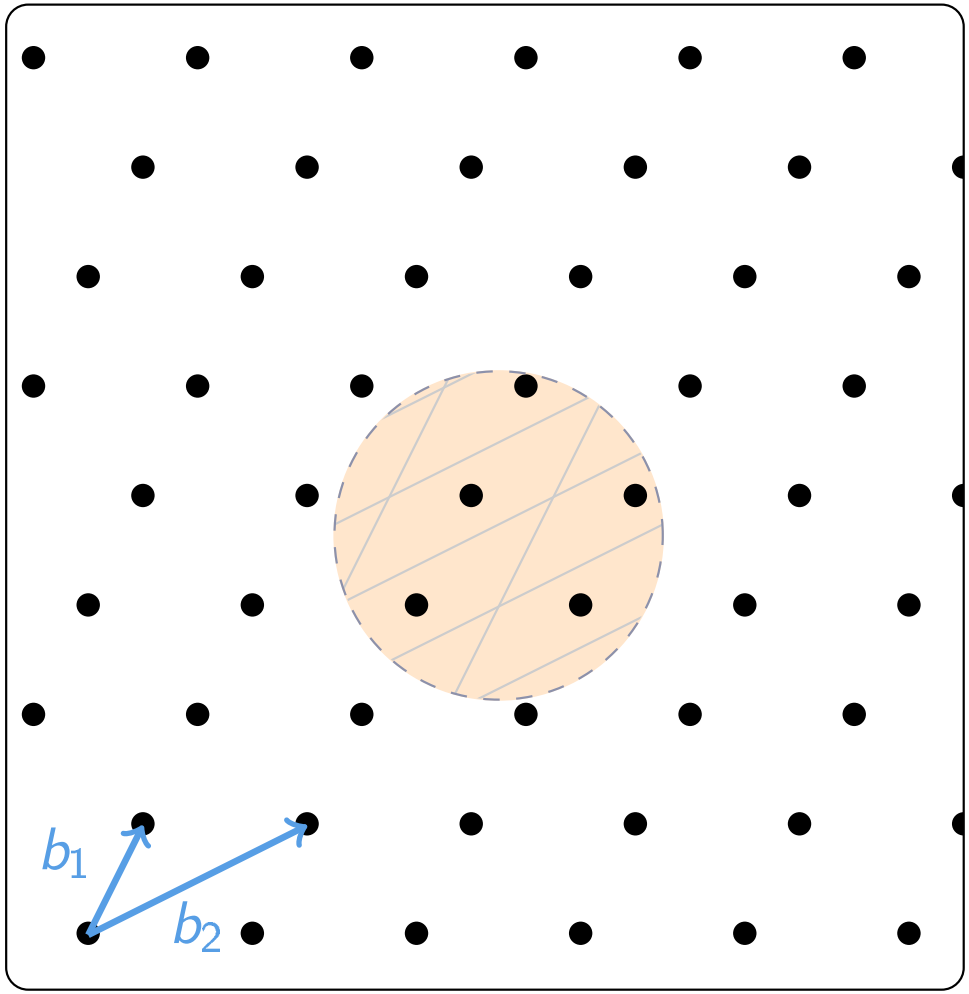

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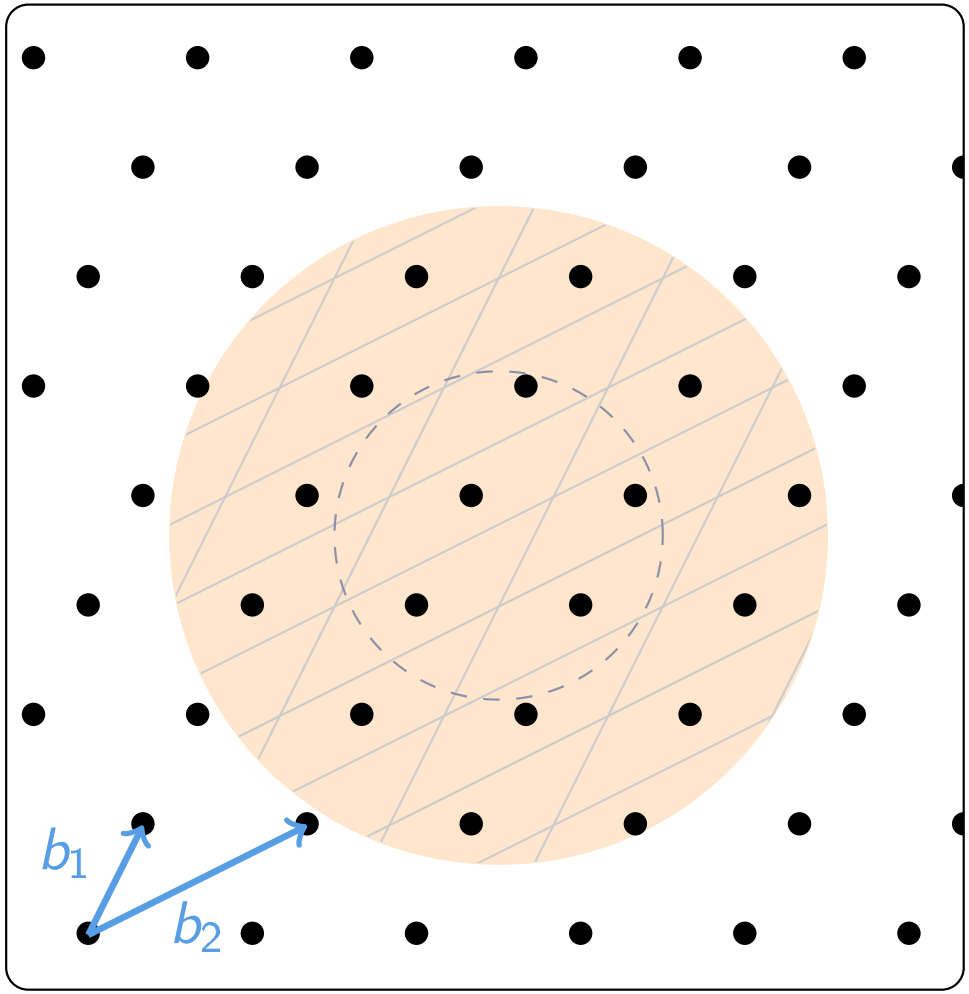

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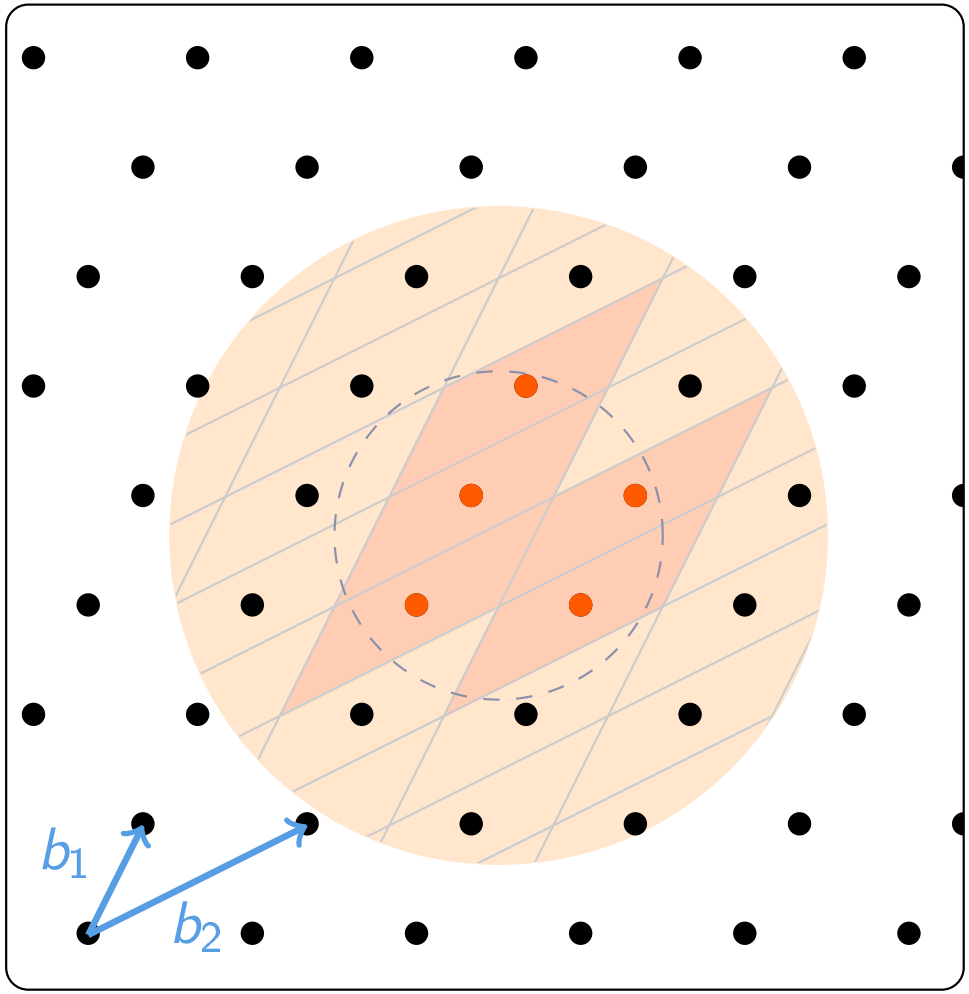

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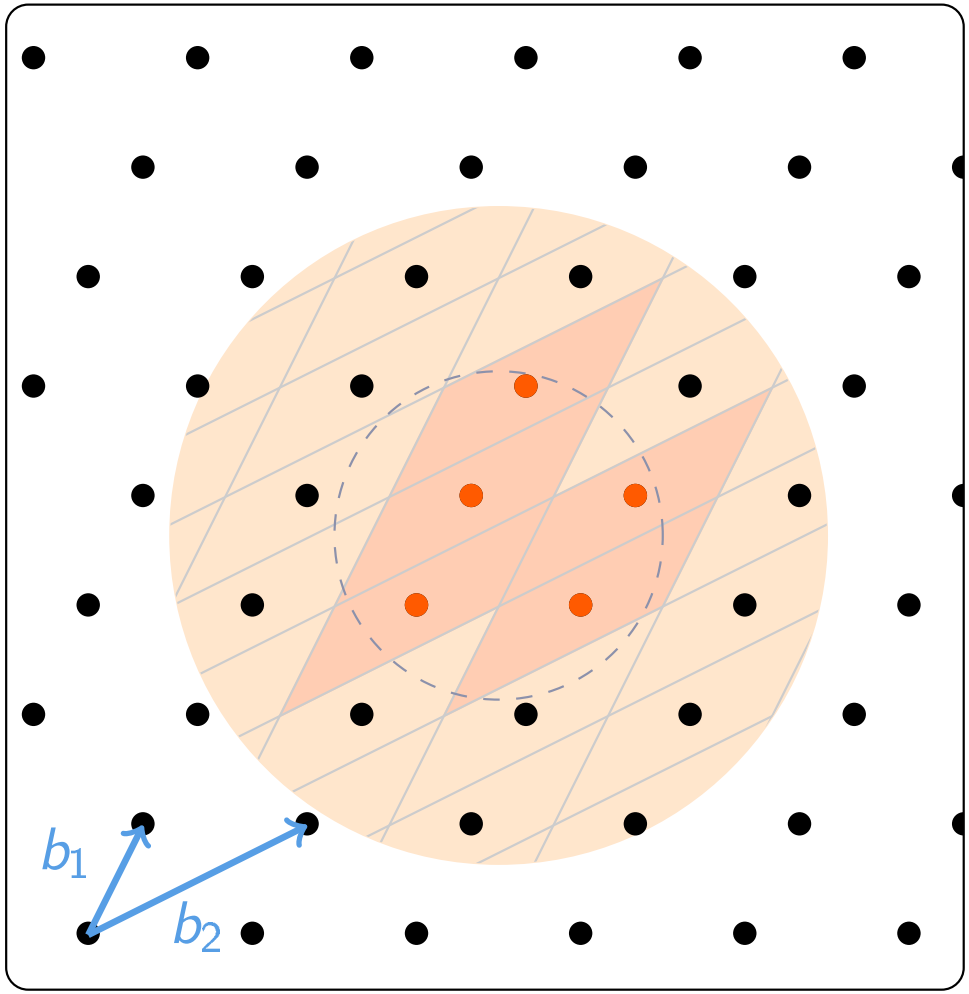

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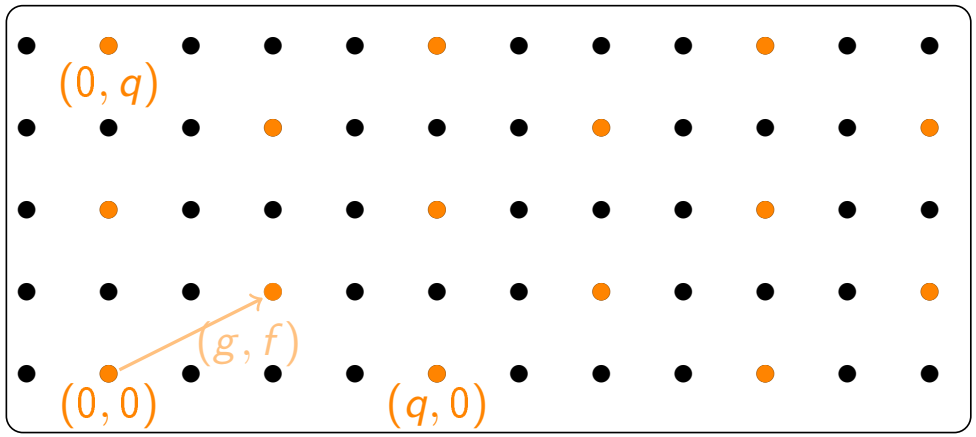
Section's conclusion

Hash-and-sign signature scheme:

- requires a lattice $\mathcal{L}$ + a short basis B_s + a bad basis B_p
- provably secure if recovering a short basis from B_p is hard

Falcon's signature scheme

(one of the three 2022 NIST standard)



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Episode 2: a random hard lattice $\mathcal{L}$ (+ a short basis) $\Rightarrow$ signature

How do we generate a hard lattice (+ a short basis)?

NTRU [HPS98]

Parameters:
- q prime and large (e.g., $q = 16411$)
- $B \ll \sqrt{q}$ integer (e.g., $B = 5$)

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- we should replace integers by polynomials (degree 512 or 1024)
- for this talk, let's keep integers (and pretend it is hard)

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

Hard lattice from NTRU assumption

NTRU assumption: an adversary cannot distinguish $h = f \cdot g^{-1} \bmod q$ (with $|f|, |g| \leq B$) from a uniform h in $\mathbb{Z}/q\mathbb{Z}$.

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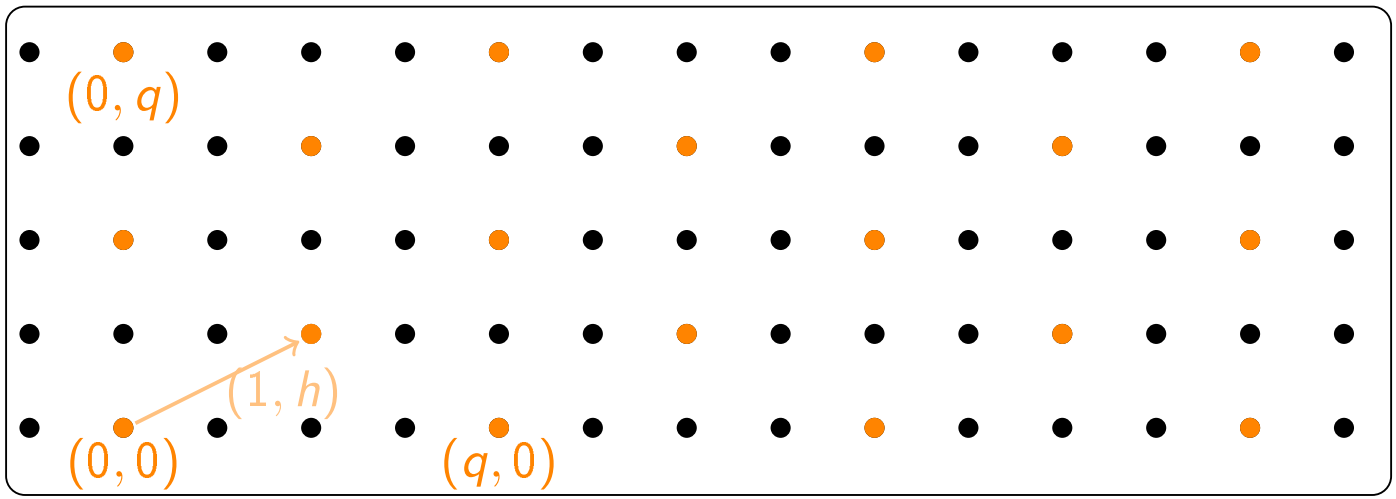

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no adversary can compute a short basis of $\mathcal{L}_h$ $\impliedby$ NTRU assumption holds

NTRU: wrapping up

We have seen: under the NTRU assumption, the following algorithm produces random hard lattices.

- sample random f, g in $\{-B, \dots, B\}$
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This is because we work with integers instead of polynomials (for simplicity)

$\Rightarrow$ if f, g and h are polynomials of degree d, $\mathcal{L}_h$ has dimension $2d$ instead of 2

Falcon signature scheme

Falcon signature:

Hash-and-sign signature

+

hard lattice from NTRU assumption

with polynomials of degree $d = 512$ or 1024 instead of integers

Falcon is one of the three post-quantum signature scheme standardized by the NIST in 2022.

[Falcon] Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Prest, Ricosset, Seiler, Whyte, Zhang. NIST standard

Falcon's performances

			Post-quantum standards (NIST 2022)		
	RSA	EdDSA	**Falcon**	Dilithium	SPHINCS+
Public key size (bytes)	256	32	**897**	1312	32
Signature size (bytes)	256	64	**666**	2420	17088
Signature time (µs)	665	51	**241**	208	35584
Verification time (µs)	19	142	**52**	74	2091

- Comparison for $\approx$ 128-bits security
- Timings for a processor of 1.6 GHz
  - RSA and EdDSA are obtained experimentally with openssl on my laptop
  - Falcon, Dilithium and SPHINCS+ are obtained from the number of cycles provided on their websites (assuming a 1.6 GHz processor)
- for SPHINCS+, this is the variant using SHAKE hash function

Summing up and open questions

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[Raccoon] del Pino, Espitau, Katsumata, Maller, Mouhartem, Prest, Rossi, Saarinen. Technical report, NIST.

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  - can lead to bugs in implementation

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Thank you